NETAJI SUBHAS OPEN UNIVERSITY

Post Graduate Degree Programme

SELF LEARNING MATERIAL

M.A. in Economics

PGEC-IV

Course : Mathematics for Economics

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in a subject is introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation. I am happy to note that the university has been recently accredited by National Assesment and Accreditation Council of India (NAAC) with grade ''A''.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing, and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the, University.

Needless to add, a great deal of these efforts are still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University

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Course : Mathematics for Economics
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Dr. Seikh Salim
Associate Professor in Economics, NSOU

Dr. Bibekananda Raychaudhuri
Associate Professor in Economics, NSOU

: Format Editor :

Mrs. Priyanthi Bagchi
Assistant Professor of Economics, NSOU

Notification

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PG : Economics (PGEC)

PGEC–IV : Mathematics for Economics

Mathematics for Economics

PGEC–IV

Unit 1 ❐ Function and its Derivative or Differentiation

Structure

1.1 Objectives

After studying this unit, the reader will be able to know

- what is a mathematical function
- what is meant by derivative or differentiation of a function
- the link between slope and curvature of a function and its derivative
- higher order partial derivatives
- the concept of total derivative
- the concept and properties of homogeneous functions
- the Euler's theorem

1.2 Introduction

Economics, generally speaking, deals with relationships among various economic variables. These relationships may concisely and precisely be discussed by the mathematical concept of 'function'. Again, while making an economic decision, we have to consider a basic question. The question is : will a particular line of action add more to our benefits than the efforts spent on the action? This is a vital question for making an economic decision or for solving any economic problem. Naturally, if benefits to be received exceed efforts to be spent, the economic decision will be undertaken. In the opposite case, the decision will be rejected. If they are equal, the matter is a case of indifference. In that case, the decision may or may not be undertaken. In a word, the answer to our basic question determines the economic viability of a line of action. In Economics, this is the core of marginal analysis which is closely related to a mathematical concept called derivative or the mathematical technique of differentiation. The mathematical concept of derivative or differentiation has made marginal analysis operative, precise and exact in economic decision making. Hence we begin our discussion on Mathematical Analysis with the notion of function and its derivative/differentiation.

1.3 Definition and Types of Functions

Simply speaking, two varibles x and y are said to be functionally related if for a particular value of x, we get a particular value of y. We generally denote the function as : $y = f(x)$. Here x is called independent variable and y is called dependent variable, and f is the functional notation stating the nature of relation between x and y. Thus $y = f(x)$ means that the value of y somehow depends on the value of x. Here the value of y depends on the value of x. Hence y is called the dependent variable and x is called the independent variable or explanatory variable. The word variable means anything whose value varies or changes.

Now, the value of y may depend on the values of a set of variables, say, x_1, x_2, ..., x_n. Then we shall write the function in usual notation as, $y = f(x_1, x_2, ..., x_n)$ where y is the dependent variable, x_1, x_2 ..., x_n are different independent variables; and f denotes the functional relationship.

Thus, technically speaking, a function is a mathematical formalisation of the relationship whereby the values of a set of independent variables determine the value of the dependent variable. Stating alternatively, a function is a mathematical relationship whereby the value of the dependent varible is determined by the values of a set of independent variables. Thus, a function is a mathematical expression of dependency between two or more variables. When we say that 'y is a function of x', it implies that for each value of x we get a single, definite value of y. So long there is a one to one correspondence between two variables, we write, y = f(x). It simply says that y changes as x does.

In this connection, two points may be noted. **First**, in the functional expression y = f(x), we have called x the independent variable while the variable y is called the dependent variable. Here, y is a function of x and it does not necesarily imply that x is also a function y. The value of x may or may not depend on the value of y. **Secondly**, in the definition of a function, we have stipulated a unique value of y for each value of x. However, the converse is not required. In other words, more than one x value may legitimately be associated with the same value of y. If there is one to one corespondence between the value of x and the value of y, we say that y is a single-valued function of x.

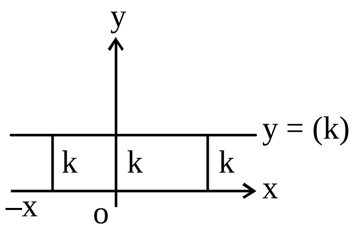
In Economics, we come across a variety of functions. If demand for a commodity (D) depends on its price (P), we write, D = f(P). If consumption (C) depends on income(Y), we may say, C = F(Y). If the rate of investment (I) depends on the rate of interest (r), we get the investment function : I = g(r). Here the symbols f, F and g all denote the functional relation in demand function, consumption function and investment function, respectively. Here, D, C, I are dependent variables while P, Y and r independent variables. If we assume that the level of saving (S) depends on the level of income (Y) and the rate of interest (r), our saving function can be written as : S = S(Y, r) where the second 'S' stands for functional relation. If the level of consimption expenditure of an economy (C) depends on the level of income (Y), the rate of interest (r), the volume of assets (a), the distribution of income (d), the age-distribution of population (A), the volume of advertisting expenditure (e), etc., the consumption function of the economy can be written in more general form as : C = f(Y, r, a, d, A, e, ...).

❒ Types of Functions

The expression y = f(x) is a general statement. A function is also called *mapping* or *transformation* which implies the action of associating one thing with another.In the statement y = f(x), the functional notation 'f' may thus be interpreted to mean a rule by which different values of x are *mapped* or *transformed* into different values of y. Thus, the expression y = f(x) implies that a mapping is possible but the actual rule of mapping is not explicitly mentioned. Depending on different rules of mapping, we get different types of functions. We mention below some specific types of functions which are more common in Economics.

❒ Constant Function :

A constant does not change its magnitude. In a given operation, it has a fixed value. For example, each number in isolation in the number system can be regarded as a constant. If we say that k is a constant such that k = 7, it implies that in any entire operation, k takes or assumes only this value of 7. Now, a function whose range consists of only one element, is called a constant function. The constant function assumes only one value or one magnitude. For example, let y = f(x) = 10 is a constant function. We can alternatively write it as y = 10 or f(x) = 10. Here the value of y remains 10 irrespective of the value of x. The general expression for a constant function is : y = k or, f(x) = k where k is a real number. In a two dimensional plane, a constant function y = k can be represented by a horizontal straight line. (Fig 1.1). In Economics, when investment (I) is autonomously given or exogeneously given at I_0, we write $I = I_0$ or, say, for example, I = ` 1000m. It is a case of constant function.



(Fig. 1.1)

Similarly, total fixed cost is fixed and does not depend on the level of output i.e., TFC = k is a constant function. Similarly, if price is fixed (as in the case of perfect competition), we write, $p = p_0$ and it is a constant function. All these are examples of constant function. In the co-ordinate plane, they will appear as horizontal straight lines.

❒ Polynomial functions

The word 'polynomial' means 'multiterm'. A polynomial equation is an equation by which, in general, several terms in an independent variable are raised to various powers. The degree of a polynomial equation is the highest power of the independent variable in that equation. A polynomial function of a single variable x has the general form :

$y = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.

This is a polynomial in x of degree n provided $a_n \neq 0$. Remembering that $x^0 = 1$ and $x^1 = x$, we may rewrite the polynomial equation as :

$y = a_0x^0 + a_1x^1 + a_2x^2 + ... + a_nx^n$

It shows a specific pattern of the equation. In this equation, each term contains a coefficient as well as a non-negative integer power of the variable x. Taking different values of the integer n, we may get several sub-categories of polynomial functions. For example, we have shown the following cases :

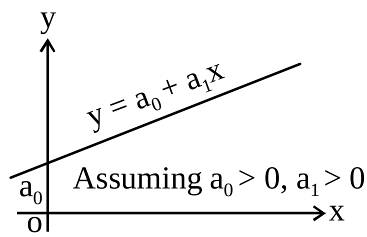
If n = 0, we have, $y = a_0$. It is a constant function

If n = 1, we have, $y = a_0 + a_1x$. It is a linear function.

If n = 2, we have, $y = a_0 + a_1x + a_2x^2$. It is a quadratic function.

If n = 3, we get, $y = a_0 + a_1x + a_2x^2 + a_3x^3$. This is a cubic function in x, and so on.

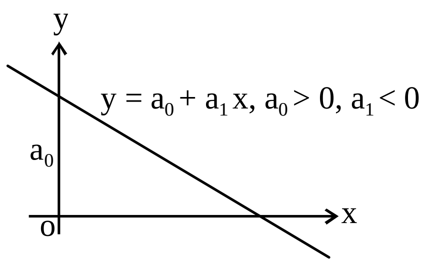
Thus, polynomial functions represent a general class of several functions. Our previous constant function is just a special case of polynomial function when the power-integer (n) is equal to zero. In other words, a constant function is a polynomial of degree zero. A linear equation is a first degree polynomial (n = 1). A quadratic equation is a second degree polynomial (n = 2). A cubic equation is a third degree polynomial (n = 3), and so on.



(Fig. 1.2)

Let us consider the shapes of these functions in a two-dimensional diagram. We have shown the shape of the constant function in our figure 1.1. A linear function will appear as a straight line when plotted in the co-ordinate plane. We have shown this in figure 1.2. Putting n = 1 in the general form of the polynomial equation, we get the first degree polynomial or the linear function : $y = a_0 + a_1x$.

In this function, a_0 is vertical intercept. (We get it by putting x = 0 in the equation). The coeffecient a_1 measures the slope (the steepness of incline) of the line. We have assumed that $a_0 > 0$ and $a_1 > 0$. As $a_0 > 0$, the straight line has a positive vertical intercept equal to a_0. As $a_1 > 0$, the straight line is upward rising. Taking example from Economics, in our consumption function, $a_0 > 0$, $a_1 > 0$. Thus, the straight line drawn in figure 1.2 resembles the shape of the consumption function. If $a_1 < 0$, the straight line will be downward sloping as shown in figure 1.3. Again, taking an example from Economics, in our linear demand function where demand (D) is an inverse function of p, we write, $D = a_0 + a_1p$, $a_1 < 0$. Thus, our straight line drawn in figure 1.3. resembles a linear demand function.

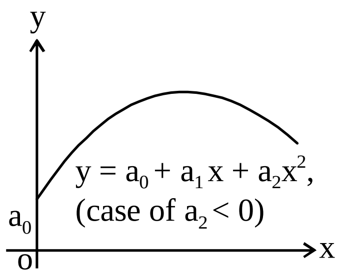


(Fig. 1.3)

A quadratic function plots as a parabola – a curve with a single bump or wiggle. Putting n = 2 in the general form of a polynomial function, we get $y = a_0 + a_1x + a_2x^2$.

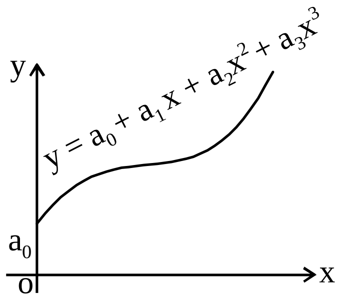
This is a quadratic function or a second degree polynomial in x. Assuming $a_0 > 0$ and $a_2 < 0$, we have drawn a quadratic function in our figure 1.4.

When $a_2 < 0$ and $a_0 = 0$, the curve will start from the origin (as a_0 is the vertical intercept of the curve) and be concave to the x-axis. For example, in Economics, our total revenue function or TR curve is generally of this shape If $a_2 > 0$, with a_0 and a_1 also positive, the curve will open the other way. It will then display, as put it by A.C. Chiang, a valley, rather than a hill. For example, our average cost (AC) and marginal cost (MC) curves resemble this shape.



(Fig. 1.4)

Let us consider the shape of a cubic function. Putting n = 3 in our general equation of the polynomial function, we get the cubic function : $y = a_0 + a_1x + a_2x^2 + a_3x^3$. This is a cubic function of x or a third degree polynomial in x. When a polynomial function of degree n is plotted on a graph paper, the number of turning points may be up to (n – 1). Thus a linear function (n = 1) has zero or no turning point. A quadratic function (n = 2) has one curvature or one turning point. Thus, a cubic function (n = 3) may have two turning points. Hence, the graph of a cubic function will, in general, manifest two wiggles. This is shown in our figure 1.5.



(Fig. 1.5)

Here we have drawn a cubic function assuming $a_2 < 0$. These functions have many uses in Economics. The curve in our figure resembles the shape of the total cost (TC) function. If we have $a_0 = 0$ and again, $a_2 < 0$, the curve will pass through the origin, keeping its shape unchanged. It will then resemble the total variable cost (TVC) curve.

❒ Rational functions

A function expressed as a ratio of two polynomial functions is known as rational function (meaning *ratio-nal*). For example, $y = \frac{3x^2 + 7x + 9}{4x + 5}$ is a rational function. As per this definition, any polynomial function must itself be a rational function. For, it can always be expressed as a ratio to 1, which is a constant function. For example, $y = 5x^2 + 3x + 6$ is a quadratic function or a polynomial of degree 2.

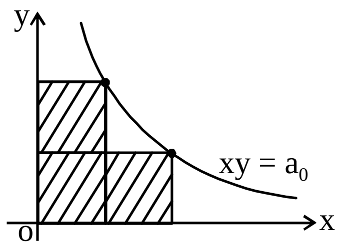
Now, $y = 5x^2 + 3x + 6 = \frac{5x^2 + 3x + 6}{1} = \frac{5x^2 + 3x + 6}{1.x^0}$

$$= \frac{\text{second degree polynomial}}{\text{constant function or zero degree polynomial}}.$$

Thus the polynomial $y = 5x^2 + 3x + 6$ is itself a rational function.

A special rational function has interesting applications in Economics. Let the function be, $y = \frac{a_0}{x}$ where a_0 is a constant (i.e. constant function).

Then, $xy = a_0$ = constant. Plotting this on a two-dimensional diagram i.e. on (x, y) plane, we get a rectangular hyperbola (Fig. 1.6). Here the product of two variables is always a constant ($xy = a_0$).



(Fig. 1.6)

This means that the area of all the rectangles obatained by joinig abscisssa and ordinate of all points on this curve is constant. Such a curve in co-ordinate geometry is called a rectangular hyperbola. Now, if x represents price and y represents quantity demanded, then xy represent total reveume of the seller or total expenditure of the buyer. Now, the equation of the rectangular hyperbola is : $xy = a_0$ = constant. So, if the total reveume of the seller or total expenditure of the buyer remains the same or constant, the demand curve will be a rectangular hyperbola. Another example from Economics is the shape of the AFC curve. We know that $AFC = \frac{TFC}{q}$.

$\therefore$ AFC × q = TFC = constant = a_0 (say)

So, plotting AFC on one axis and output (q) on the other, the AFC curve will be a rectangular hyperbola.

The rectangular drawn from $xy = a_0$ never meets the axes. Rather the curve approaches the axes asymptotically. As y becomes very large, x will become very small, but not equal to zero i.e., the curve will not meet the y-axis. Similarly, if x becomes very large, y will be very small, but not equal to zero, i.e., the curve will not meet the x-axis either. In symbols, as $y \to \propto$, $x \to 0$ and as $x \to \propto$, $y \to 0$. Such a curve is generally referred to as an asymptotic curve.

❐ Inverse function

We know that a function y = f(x) represents a one-to-one correspondence or one-to-one mapping. This means that for a particular value of x, we get a particular value of x. Now, the function y = f(x) may have an inverse function, say, $x = f^{-1}(y)$. It is read as 'x is an inverse function of y'. Here, f^{-1} represents a functional symbol. It does not mean the reciprocal of the function f(x). Thus, $x = f^{-1}(y) = h(y)$ (say). Thus, the symbol f^{-1}

signifies a function related to the function f. For example, y = 10x + 7, then $x = \frac{1}{10}(x-7)$. These two are inverse functions of each other. If y = f(x) = 3x, then, alternatively, $x = \frac{1}{3}$ y = h(y).

Let us give an example of an inverse function from Economics. Let quantity demanded (q) be the function of price : q = f(p). Let the function be linear. Let its specific form be : $q = \frac{a}{b} - \frac{1}{b}.p$. (a > 0, b > 0). Plotting q on the vertical axis and p on the horizontal axis, we get a downward sloping linear demand function. Let us deduce the inverse demand function from this demand function. We may write, $\frac{1}{b}p = \frac{a}{b} - q$, or, p = a – bq. This is the inverse demand function, say, p = h(q) of our previous demand function. Plotting p on the vertical axis and q on the horizontal axis, we get, once again, a downward sloping demand function. By demand function we generally mean this inverse demand function proper.

❒ Non-algebraic Function

Any function expressed in terms of polynomials and/or roots, such as, square root of polynomials is an algebraic function. So far, functions we have discussed are all algebraic functions. If, however, the independent variable does not appear as a polynomial, the function is said to be a non-algebraic function. It may be of three types :

(i) Exponential function, for example, $y = ab^x$.

(ii) Logarithmic function, for example, $y = \log_b x$

(iii) Trigonometric function, for example, y = cos x.

Trigonometric functions are also called circular functions.

Non-algebraic functions are also known by the more esoteric name of transcendental functions.

1.4 Concepts of Derivative and Differentiation

When two variables x and y are somehow related, we express that relation by the functional notation, say, y = f(x). It simply states that the value y depends on the value of x. In other words, it states that the value of y changes as the value of x does. Now, suppose when $x = x_0$, $y = f(x_0) = y_0$(say). Further, suppose that x changes its value from the initial value of x_0 to x_1 and correspondingly the value of y also changes from its initial value of y_0 to y_1. Then the rate of change of y due to change in x is equal to $\frac{y_1 - y_0}{x_1 - x_0}$. The concept of derivative gives us the rate of change of the dependent variable when the change in the independent variable is very small. If we denote the change in y by Δy i.e., $\Delta y = y_1 - y_0$ and the change in x by Δx i.e., $\Delta x = x_1 - x_0$, then we can write, $\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$. This is the change in y per unit change in x. Now, if the change in x is very small ($\Delta x \to 0$), we call it derivative of y with respect to x. It is denoted by $\frac{dy}{dx}$.

Thus, $\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. Thus, derivative of a function gives us an idea about the rate of change of the dependent variable when the independent variable changes by a very small amount ($\Delta x \to 0$). Thus, the derivative of y function, $\frac{dy}{dx}$ is change in y due to infinitesimal change in x. The act of finding the value of this derivative $\frac{dy}{dx}$ is called differentiation.

Let us see how this value can be found out. We have said that y = f(x). Initially, when $x = x_0$, $y = f(x_0) = y_0$(say) and as x changes to $x_1(= x_0 + \Delta x)$, y changes to $f(x_1) = y_1$(say).

So, we can write, $\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$.

We have mentioned that x changes by Δx amount. So, the new value of x i.e., $x_1 = x_0 + \Delta x$. Hence we can write, $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ as $x_1 - x_0 = \Delta x$. We generally use h for Δx.

So, $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$. This is the rate of change of y at a given value of x (say, x_0) and is known as instanteneous rate of change. In general we can write for any value of x, $\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}$.

The derivative of y = f(x) is obtained when $\Delta x(= h)$ tends to zero. This derivative of y = f(x) with respect to x is generally denoted by $\frac{dy}{dx}$ or f′(x) or $\frac{d}{dx}(y)$ or $\frac{d}{dx}[f(x)]$.

Thus, $\frac{dy}{dx} \equiv f'(x) \equiv \frac{d}{dx}(y) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$. To repeat, the act of finding out the value of $\frac{dy}{dx}$ or the value of derivative is called differentiation. The method is known as differentiation from first principle or differentiation from definition. It may be mentioned in this connection that $\frac{dy}{dx}$ is not a ratio of dy to dx. Rather, it indicates an operation– an operation of finding out the value of $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$. The alternative notation f′(x) or $\frac{dy}{dx}$ explicitly reflects this idea. Let us give some examples of finding out the value of $\frac{dy}{dx}$ from the function y = f(x).

Example 1.1 : Given y = 10x + 7, find $\frac{dy}{dx}$.

Solution : Here, y = f(x) = 10x + 7.

∴ f(x + h) = 10(x + h) + 7

Hence, $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{10(x+h)+7-(10x+7)}{h} = 10$

So, $\underset{h \to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h} = \underset{h \to 0}{\text{Lt}}\, 10 = 10$

So, $\frac{dy}{dx} = f'(x) = 10$

Here y = 10x + 7 and it is a linear function in x. We see that the derivative of a linear function is constant and it is equal to the gradient or slope of the straight line. Here the derivative is positive (+10). It indicates that both x and y change in the same direction.

Example 1.2. : Given y = 7 – 8x, find $\frac{dy}{dx}$.

Solution : Here y = f(x) = 7 – 8x. It is again a linear function in x. Here y is an inverse function of x. We have y = f(x) = 7 – 8x. Now, x changes by Δx. Let Δx = h.

So, f(x + h) = 7 – 8(x + h).

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{7-8(x+h)-(7-8x)}{h} = -8$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{h \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = -8 \text{. Thus, } \frac{dy}{dx} = f'(x) = -8$$

Here the derivate is negative (= –8). Thus, in this case, the gradient or slope of the straight line is negative. A negative sign of the derivative implies that the independent and the dependent variables change in the opposite directions.

Our examples (1) and (2) show that the derivative of a linear function is constant and it is equal to the slope or gradient of the straight line. We may prove it by taking a general equation in linear form. This we have done in example (3) below.

Example 3 : y = mx + c (m $\gtrless$ 0, c $\gtrless$ 0). Find $\frac{dy}{dx}$.

Solution : We know that $\frac{dy}{dx} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$

Here, y = f(x) = mx + c (m $\gtrless$ 0, c $\gtrless$ 0)

∴ f(x + h) = m(x + h) + c

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{m(x+h)+c-(mx+c)}{h} = m \text{. So, } \frac{dy}{dx} = \underset{h \to 0}{\text{Lt}}\, m = m$$

Thus, the derivative of a linear function is constant and it is equal to the gradient or slope (positive or negative) of the linear function.

Example 1.4 : Determine the derivative of the function y = $3x^2$ + 5x + 6

Solution : Here y = f(x) = $3x^2$ + 5x + 6

∴ f(x + h) = $3(x + h)^2$ + 5(x + h) + 6

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2+5(x+h)+6-(3x^2+5x+6)}{h}$$

$$= \frac{6xh+3h^2+5h}{h} = \frac{h(6x+3h+5)}{h} = 6x + 3h + 5$$

$$\text{Now, } \frac{dy}{dx} = \underset{h \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{h \to 0}{\text{Lt}}(6x+3h+5) = 6x + 5$$

Note that here $\frac{dy}{dx} = f'(x)$ is a function of x. If x = 1, f′(1) = 11; if x = 2, f′(2) = 17, etc.

Example 1.5. : Obtain the derivative $\frac{dy}{dx}$ of the function $y = x^3$

Solution : Here $y = f(x) = x^3$

$\therefore$ $f(x + h) = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

Now, $\frac{\Delta y}{\Delta x} = \frac{f(x+a)-f(x)}{h} = \frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$

$= \frac{3x^2h+3xh^2+h^3}{h} = 3x^2 + 3xh + h^2$

Now, $\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{h\to 0}{\text{Lt}}(3x^2+3xh+h^2) = 3x^2.$

So, $\frac{dy}{dx}$ of the function $y = x^3$ is $3x^2$.

Here also, $\frac{dy}{dx}$ or f′(x) is a function of x, i.e., f′(x) varies with the variation in the value of x. If x = 1, f′(1) = 3. If x = 2, f′(2) = 12. If x = 3, f′(3) = 27 and so on.

Example 1.6. : Obtain $\frac{dy}{dx}$ when $y = ax^2 + bx + c$

Solution : Here y is a quadratic function of x or a second degree polynomial in x.

Now, $\frac{dy}{dx} = \underset{\Delta x\to 0}{\text{Lt}} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. Putting $\Delta x = h$,

we may write, $\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$.

Now, $y = f(x) = ax^2 + bx + c$

$\therefore$ $f(x + h) = a(x + h)^2 + b(x + h) + c$

$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$

$= \frac{a(x+h)^2+b(x+h)+c-(ax^2+bx+c)}{h}$

$$= \frac{2axh + bh + h^2}{h} = 2ax + b + h$$

Now, $\frac{dy}{dx} = \underset{h \to 0}{Lt}(2ax + b + h) = 2ax + b$

1.5 Rules of Differentiation / Rules of Derivative

There are some rules which can help us to find out derivative of a function. We here do not offer any proof of those rules. We are just stating those rules which can be applied only technically to determine the derivative of a function. Some of such important rules are mentioned below.

Rule 1 : If y = c where c is a constant, $\frac{dy}{dx} = 0$. This is known as the rule of differentiation of a constant function. A constant does not depend on any variable. So, if x changes by dx amount, y does not change i.e., dy = 0. Hence, $\frac{dy}{dx} = 0$

If y = 50(say), then $\frac{dy}{dx} = 0$. If $y = y_0$, $\frac{dy}{dx} = 0$

Taking example from Economics, the total fixed cost(TFC) of a firm in the short run does not depend on the level of output(q). TFC remains fixed. So, TFC = a_0(say). So, $\frac{dTFC}{dq} = 0$

Rule 2 : If $y = ax^n$ (where a and n are constants), $\frac{dy}{dx} = n\,ax^{n-1}$. This is known as the Rule of Differentiation of a power function.

Examples : (i) If $y = 10x^4$, $\frac{dy}{dx} = 4 \times 10.x^{4-1} = 40x^3$

(ii) If $y = 5x^{10}$, $\frac{dy}{dx} = 10 \times 5.x^{10-1} = 50x^9$.

(iii) If $y = x = x^1$, $\frac{dy}{dx} = 1.x^{1-1} = x^0 = 1$

(iv) If y = 30x, $\frac{dy}{dx} = 1.30.x^{1-1} = 1 \times 30 \times 1 = 30$

(v) If $y = \frac{10}{x^7} = 10x^{-7}$, $\frac{dy}{dx} = -7 \times 10.x^{-7-1} = -70x^{-8} = -\frac{70}{x^8}$

(vi) If $y = \frac{1}{x} = x^{-1}$, $\frac{dy}{dx} = -1.x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$, etc.

It may be noted that the earlier result of differentiation of a constant function can be obtained from this rule of differentiation of a power function. Let y = c where c is constant. To apply the rule of differentiation of a power function, we rewrite the value of y as : $y = c.x^0$ (as $x^0 = 1$)

Now, $\frac{dy}{dx} = 0 \times c.x^{0-1} = 0$, a result which we have already stated in Rule 1.

Rule 3 : Sum or Difference Rule of Differentiation

If $y = u \pm v$ where both u and v are functions of x, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

Example 1.7 :

(i) $y = 10x^3 + 7x^5$. Then $\frac{dy}{dx} = 3 \times 10.x^{3-1} + 5 \times 7.x^{5-1} = 30x^2 + 35x^4$

(ii) $y = \frac{30}{x} + \frac{40}{x^3}$. We have to find out $\frac{dy}{dx}$. The given function can be re-written as, $y = 30x^{-1} + 40x^{-3}$

Now, $\frac{dy}{dx} = -1 \times 30x^{-1-1} + (-3).40.x^{-3-1} = -30x^{-2} - 120x^{-4} = -\frac{30}{x^2} - \frac{120}{x^4}$

(iii) $y = 50x^3 - 70x^2$

$$\frac{dy}{dx} = 3 \times 50.x^{3-1} - 70 \times 2.x^{2-1} = 150x^2 - 140x$$

(iv) $y = -\frac{20}{x^2} - \frac{50}{x^5}$

To calculate $\frac{dy}{dx}$ of the function, we rewrite the function as, $y = -20x^{-2} - 50x^{-5}$.

Now we apply the rule of differentiation of power function.

$$\frac{dy}{dx} = -2(-20)x^{-2-1} - (-5 \times 50).x^{-5-1} = 40x^{-3} + 250x^{-6} = \frac{40}{x^3} + \frac{250}{x^6}$$

(v) If $y = ax^2 + bx + c$ where a, b and c are positive or negative constants,

$$\frac{dy}{dx} = 2ax + b$$

(vi) If $y = a_0 + a_1x + a_2x^2 + a_3x^3$ where a_0, a_1, a_2 and a_3 are constants,

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2.$$

In general, if y is polynomial of degree n, we have, $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ (where a_0, a_1, a_2 etc. are positive or negative constants). Then $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + (n - 1)\, a_{n-1}x^{n-2} + na_nx^{n-1}$.

For example, if $y = 7x^5 - 4x^4 - 3x^3 + 4x^2 + 8x + 10 - \frac{7}{x} + \frac{10}{x^2} - \frac{20}{x^3}$,

$$\frac{dy}{dx} = 35x^4 - 16x^3 - 9x^2 + 8x + 8 + \frac{7}{x^2} - \frac{20}{x^3} + \frac{60}{x^4}$$

Thus, we see that the derivative of the sum (or difference) of two or more functions is actually the sum (or difference) of the derivatives of two or more functions.

Rule 4 : If y = u.v where u and v both are functions of x, $\frac{dy}{dx} = \frac{du}{dx}.v + \frac{dv}{dx}.u$. This is known as product rule of differentiation.

Examples :

(i) $y = (3x^4 + 5x^2)(10x^2 + 3x + 9)$

$$\frac{dy}{dx} = (12x^3 + 10x)(10x^2 + 3x + 9) + (20x + 3)(3x^4 + 5x^2)$$

(ii) $y = (5x^2 - 4x)(7x^3 - 3x^2 + 4x - 6)$

$$\frac{dy}{dx} = (10x - 4)(7x^3 - 3x^2 + 4x - 6) + (21x^2 - 6x + 4)(5x^2 - 4x)$$

Rule 5 : If $y = \frac{u}{v}$ where both u and v are functions of x, then $\frac{dy}{dx} = \frac{\frac{du}{dx}.v - \frac{dv}{dx}.u}{v^2}$

This is known as quotient rule of differentiation.

Example 1.8 :

(i) $\frac{7x^2 + 5}{9x^2 + 2x + 8}$, find $\frac{dy}{dx}$.

Let u = $7x^2 + 5$ and ν = $9x^2 + 2x + 8$

$\therefore \frac{du}{dx} = 14x$ and $\frac{dv}{dx} = 18x + 2$

Now, if $y = \frac{u}{\nu}$, then $\frac{dy}{dx} = \frac{\frac{du}{dx}.\nu - \frac{d\nu}{dx}.u}{\nu^2}$

So, $\frac{dy}{dx} = \frac{14x(9x^2+2x+8)-(18x+2)(7x^2+5)}{(9x^2+2x+8)^2}$

(ii) Let $y = \frac{5x^3+7}{2x^2+4x+3}$. Find $\frac{dy}{dx}$.

Here $\frac{dy}{dx} = \frac{(2x^2+4x+3)\frac{d}{dx}(5x^3+7)-(5x^3+7)\frac{d}{dx}(2x^2+4x+3)}{(2x^2+4x+3)^2}$

$= \frac{(2x^2+4x+3)(15x^2)-(5x^3+7)(4x+4)}{(2x^2+4x+3)^2}$

(iii) Let $y = \frac{3x^2+7}{2x+1}$. Find $\frac{dy}{dx}$.

Here $\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(3x^2+7)-(3x^2+7)\frac{d}{dx}(2x+1)}{(2x+1)^2}$

$\therefore \frac{dy}{dx} = \frac{(2x+1)\ 6x-(3x^2+7)\times 2}{(2x+1)^2} = \frac{12x^2+6x-6x^2-14}{4x^2+4x+1} = \frac{6x^2+6x-14}{4x^2+4x+1}$

Rule 6 : If y = f(z) and z = F(x), then $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$. This is known is chain rule of differentiation or function of a function rule. Let us check the rule.

Let $y = z^2$ and again, z = 2x + 1. We can, in this case, directly find out $\frac{dy}{dx}$ by expressing y as a function of x and then diffentiating y with respect to x.

Here $y = z^2 = (2x + 1)^2 = 4x^2 + 4x + 1$

$\therefore \frac{dy}{dx} = 2 \times 4x^{2-1} + 4.x^{1-1} = 8x + 4.$

Let us see what happens if we apply the chain rule. We have $y = z^2$ and $z = 2x + 1$.

So, $\frac{dy}{dz} = 2z$ and $\frac{dz}{dx} = 2$

Now, as per chain rule of differentiation, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 2.z \times 2 = 4z$

Putting $z = 2x + 1$, we get, $\frac{dy}{dx} = 4(2x+1) = 8x+4$

This is our earlier result by direct method.

Rule 7 : If $y = \log x$, $\frac{dy}{dx} = \frac{1}{x}$

Rule 8 : If $y = e^x$, $\frac{dy}{dx} = e^x$

Corollary : If $y = e^{mx}$, then $\frac{dy}{dx} = e^{mx}.\frac{d}{dx}(mx) = e^{mx}.m = me^{mx}$

Thus, if $y = e^{3x}$, then $\frac{dy}{dx} = 3e^{3x}$

Derivative of an inverse function

If $y = f(x)$ and its inverse function is : $x = g(y)$ then $\frac{dy}{dx}.\frac{dx}{dy} = f'(x).g'(y)=1$

So, $\frac{dy}{dx}$ or $f'(x) = \frac{1}{g'(y)}$ and $\frac{dx}{dy}$ or $g'(y) = \frac{1}{f'(x)}$

1.6 Concept of Higher Order Derivatives or Higher Order Differentiation

Let $y = f(x)$. Then $\frac{dy}{dx}$ or $f'(x)$ is the derivative, or more specifically, the first derivative of the function. If we differentiate this derivative again, we get the second derivative. It is denoted by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$. Similarly, we may get, if possible, third derivative $\left(\frac{d^3y}{dx^3}\right)$, fourth derivative $\left(\frac{d^4y}{dx^4}\right)$, etc. These are called higher derivatives.

Let us see how many times a polynomial can be differentiated. Suppose, y = 3x + 1. It is a polynomial of degree 1 or first degree polynomial. In this case, $\frac{dy}{dx} = 3$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$. It cannot be differentiated further. So, a first degree polynomial can be differentiated twice. Consider a second degree polynomial, say, $y = 2x^2 + 5x + 6$. Here $\frac{dy}{dx} = 4x + 5$, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 4$ and $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 0$. The function can not be differentiated for still higher order. Thus for a second degree polynomial, we may get derivative up to third degree. Consider a third degree polynomial, $y = x^3 + 9$. Then, $\frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$, $\frac{d^3y}{dx^3} = 6$ and $\frac{d^4y}{dx^4} = 0$ Thus, a third degree polynomial can be differentiated maximum 4 times. In general, an n-th degree polynomial can be differentiated (n + 1) times i.e., we may get up to (n + 1)-st order derivative.

Consider another possibility. Let $y = \frac{k}{x}$ where k is a constant (Note that this is not a first degree polynomial. It can be written as, $y = kx^{-1}$ – the value of power of x is not 1, rather minus one).

Now in this case, $\frac{dy}{dx} = -\frac{k}{x^2}$, $\frac{d^2y}{dx^2} = 2k.x^{-3} = \frac{2k}{x^3}$, $\frac{d^3y}{dx^3} = -6k.x^{-3-1} = -6kx^{-4} = -\frac{6k}{x^4}$, $\frac{d^4y}{dx^4} = 24.k.x^{-4-1} = 24k.x^{-5} = \frac{24k}{x^5}$ and so on. Here we can differentiate the function for any order.

We have noted that higher order derivatives can be obtained by the same rule of differentiation. Another point should also be noted. For the function y = f(x), $\frac{dy}{dx}$ or $f'(x)$ gives us the rate of change of y with respect to x. Similarly, its second order derivative $\frac{d^2y}{dx^2}$ or, say, $f''(x)$ gives the rate of change of $\frac{dy}{dx}$ or $f'(x)$ with respect to x. Similar explanation may be given for further higher order derivatives.

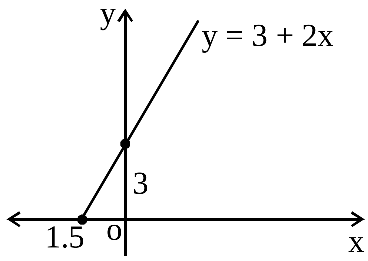
1.7 Slope and Curvature

We first consider the slope of a funcion. Let y = f(x) be a function. It may either be linear or non-linear, depending on the nature of the function. The slope or gradient of a function at any point is the first order derivative of the function, i.e., $\frac{dy}{dx}$ at that point.

1.7.1 Slope of a Linear Function

Sometimes the graphical presentation of a function may be linear. Then the slope of the linear function will be the tan of the angle between the curve and the horizontal axis on its positive direction. If that angle is θ, then the slope of the linear function = tan $\theta = \frac{dy}{dx}$.

Example : Let the specific form of the function y = f(x) be y = 3 + 2x. This is a linear function of x of the form : y = mx + c. Here, m = 2 and c = 3. Its slope = $\frac{dy}{dx}$ = m. In our specific linear function, slope = $\frac{dy}{dx}$ = m = tan θ = 2. Its vertical intercept = 3 which is obtained by putting x = 0. If x = 0, then y = 3 + 2x = 3. In the figure 1.7, we have drawn the function : y = 2x + 3. We should note that if x changes by 1 unit, y changes by 2 units. If x = 1, y = 5; if x = 2, y = 7; if x = 3, y = 9, etc. Thus, the slople of a linear function gives us the change in dependent variable if independent variable changes by one unit. If the slope is positive, the independent variable and the dependent variable move in the same direction.

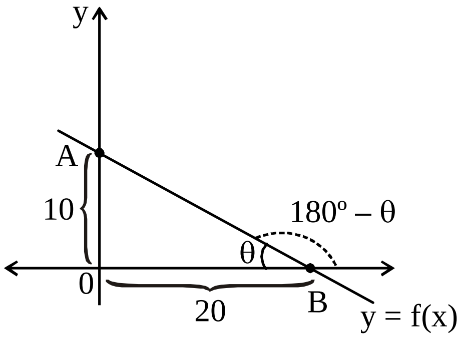


(Fig. 1.7)

In our example, if x rises by 1 unit, y will also rise by 2 units, and if x falls by 1 unit, y will also fall by 2 units.

We may cite an example from Economics. Suppose our consumption function is : C = a + bY where C = consumption, Y = Income. a and b are constants (a > 0, b > 0). We assume that 0 < b < 1. Here, slope = $\frac{dC}{dY}$ = tan θ = b. It implies that if income (Y) rises by 1 unit, consumption(C) rises by b units. In Economics, $b\left(=\frac{dC}{dY}\right)$ is called the marginal propensity to consume (MPC).

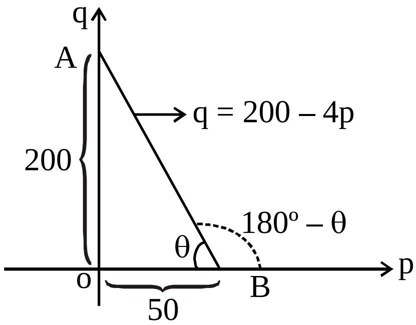
So far we have assumed that the function y = f(x) is positively sloped i.e., its $\frac{dy}{dx}$ or $f'(x) > 0$. It means that x and y change in the same direction. However, it may happen that there is an inverse relationship between x and y. In that case, if x rises, y will fall and *vice versa*. Then the function will slope downward from left to right. Let us assume that the slope of the function is negative or the function is negatively sloped and it is linear. If a function y = g(x) is represented by a downward sloping straight line, then the slope of the function is negative. For example, let $y = g(x) = 10 - \frac{1}{2}x$.



(Fig. 1.8)

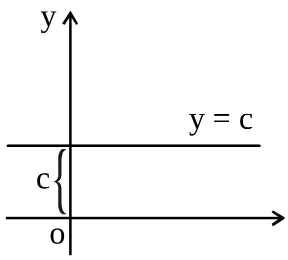
Here, vertical (y) intercept = 10 and slope = $\frac{dy}{dx} = -\frac{1}{2} < 0$. Graphically it will be a downward sloping straight line with vertical (y) intercept = 10 units and horizontal (x) intercept = 20 units. In terms of our figure, slope = tan (180º – θ) = – tan θ = $-\frac{OA}{OB} = -\frac{10}{20} = -\frac{1}{2}$. It implies that in this case, if x rises by 1 unit, y will fall by $\frac{1}{2}$ unit and *vice versa*.

We cite an example from Economics. The law of demand states an inverse relation between price (p) and quantity demanded (q). So, q = f(p) such that $\frac{dq}{dp} = f'(p) < 0$. Let the specific equation of the demand function be, q = 200 – 4p. It is a downward sloping straight line with quantity(q)- intercept = 200. Here slope = $\frac{dq}{dp} = -4 < 0$. Graphically, slope = tan (180º – θ) = – tan θ = $-\frac{OA}{OB} = -\frac{200}{50} = -4$. It implies that if p rices by one unit, quantity demanded(q) will fall by 4 units and *vice versa*. In the figure 1.9, we have drawn the specific demand function : q = 200 – 4p.

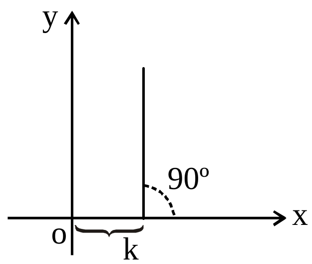


(Fig. 1.9)

We may mention the value of slope of a function under two special cases. If y = c where c is a constant it can be represented by a horizontal straight line with a vertical (y) intercept = c. Its slope = $\frac{dy}{dx} = 0$ as dy = 0, dx ≠ 0. Graphically, slope = tan θ = tan 0° = 0 (fig. 1.10). If x = k where k is a constant, its can be represented by a vertical straight line. Its horizontal(x) intercept will be k (fig. 1.11). It slope = $\frac{dy}{dx} = \infty$ (as dx = 0 and dy ≠ 0). Graphically, slope of this vertical straight line = tan θ = tan 90° = ∞.



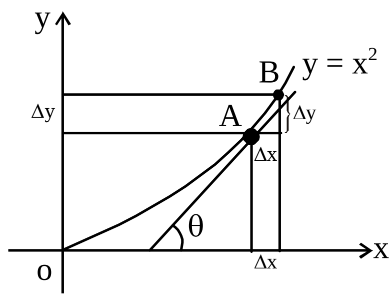
(Fig. 1.10)



(Fig. 1.11)

1.7.2 Slope of a Non-linear Function

So far we have considered slope of a linear function. Let us consider the slope of a curve. Let y = f(x) = x^2. It represents a curve. In figure 1.12, we have shown the shape of this curve taking only the positive values of x. Slope of a curve at every point is different. However, slope of a straight line is the same at all points. In the case of a non-linear function or curve, slope at any point is equal to the slope of the tangent drawn at that point. Hence, in our figure 1.12, slope of the curve at A = slope of the portion AB where AB is very very small (AB → 0) = $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ = slope of the tangent at A = tan θ where θ is the angle between the tangent and the x axis on its positive direction. Here slope of the curve is positive in the first quadrant (x > 0).

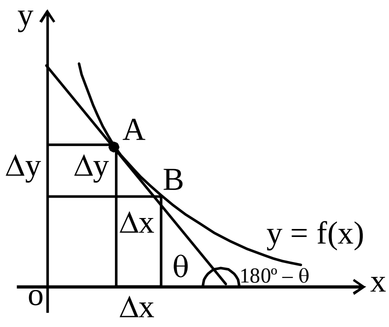


(Fig. 1.12)

Now, suppose the function y = f(x) represents a non-linear inverse relation between x and y. In this case, the function depicts a downward sloping curve as shown in the figure 1.13. Here the slope of the curve is negative. Its slope of the portion AB =

= slope of the tangent at A = tan(180° – θ) = – tan θ. Here slope of the curve is negative. We take a simple example. Let y = f(x) = $\frac{k}{x}$ where k is a positive constant (k > 0).

Then, xy = k and we know that it can be represented by a rectangular hyperbola. Now the slope of this function at any point = $\frac{dy}{dx} = -\frac{k}{x^2} < 0$, i.e., the slope of the given curve is negative.



(Fig. 1.13)

1.7.3 Curvature of a Function

In order to know the curvature of a function, we have to consider the change in its slope i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or, $\frac{d^2y}{dx^2}$. A linear function or straight line has no curvature. Its slope is constant and so, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2} = 0$. Again, consider the curvature of the curve drawn in the figure 1.12. If we go from left to right on this curve, its steepness rises. If we draw tangent at different points on this curve going from left to right, the tangents will be steeper and steeper. The equation of the curve was : y = x^2. So, its slope at any point = $\frac{dy}{dx} = 2x$. Thus, the slope of the function depends on the value of x (i.e., a function of x). As x rises, slope rises and *vice versa*. Mathematically speaking, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ = 2 > 0. This implies that $\frac{dy}{dx}$ or slope rises. Here the curve is convex upward. Again, in the figure 1.13, the slope of the curve is negative. The equation of the curve is : y = $\frac{k}{x}$ and hence $\frac{dy}{dx} = -\frac{k}{x^2} < 0$. As x rises, the absolute slope of the curve falls. Considering the negative magnitude we shall say that its slope rises as x rises. Mathematically speaking, change in slope = $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ = 2k.x^{-2-1} = $\frac{2k}{x^3}$. Thus, for x > 0, $\frac{d^2y}{dx^2} > 0$ i.e., the negative slope rises or absolute slope falls. Here the curve is convex downward.

We may mention the following cases of curvature of the function y = f(x). We here consider the case when x rises through 'a' and the shape of curve at x = a.

A. Cases of an upward rising function

(i) $f'(a) > 0$, $f''(a) = 0$, the curve is upward rising linear.
(ii) $f'(a) > 0$, $f''(a) > 0$, the curve is upward rising convex.
(iii) $f'(a) > 0$, $f''(a) < 0$, the curve is upward rising concave.

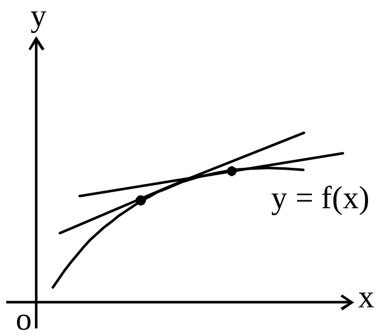
B. Cases of a downward sloping function

(iv) $f'(a) < 0$, $f''(a) = 0$, the curve is downward sloping linear.
(v) $f'(a) < 0$, $f''(a) > 0$, the curve is downward sloping convex.
(vi) $f'(a) > 0$, $f''(a) < 0$, the curve is downward sloping concave.

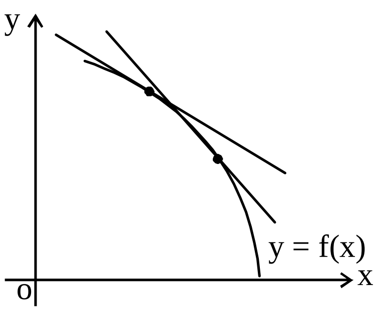
We have shown the shape of the curve of case (i) in the figure 1.7, the case of (ii) in the figure 1.12, the case of (iv) in the figure 1.8 and the case of (v) in the figure 1.13. The shapes of the curve in case (iii) is shown in the figure 1.14.

Here the curve is upward rising concave. We see that as x increases, tangents become flatter and flatter, though tangents are upward rising. Thus, slope of the curve is positive but diminishing. Here the curve is said to be upward rising concave.

The case of (ii) i.e., the shape of a downward sloping concave curve has been shown in the figure 1.15.



(Fig. 1.14)



(Fig. 1.15)

Here we see that tangents at different points on the curve is negatively sloped. So, the slope of the curve y = f(x) is negative. Again, as x rises i.e., as we move from left to right, the tangents become steeper and steeper or absolute slope rises. But here slope is negative. So we shall say that slope falls, i.e., $\frac{d^2y}{dx^2} < 0$. In this case, the curve y = f(x) is said to be downward sloping concave.

1.8 Multivariate Functions and their Derivatives

A function having more than one independent variable is called multivariate function. If the independent variables are x_1, x_2 ... x_n, then the general form of a multivariate function is : $y = f(x_1, x_2, ..., x_n)$.

In our previous discussions on function, we took a single independent variable x and the function was written as, $y = f(x)$. That function may be called univariate function as the number of independent variable in this case is unity. If the number of independent variables is two, it is called bivariate function. A bivariate function is a special case of multivariate function where the number of independent variable is two. It is written as, $y = f(x_1, x_2)$ where x_1 and x_2 are two independent variables and y is the dependent variable.

Differentiation of a multivariate function, say, $y = f(x_1, x_2)$ can produce three types of derivatives, namely, partial derivative, total derivative and total differential. We explain them one by one.

1.8.1 Partial Derivative :

Suppose we have a multivariate function $y = f(x_1, x_2)$. Now, it may happen that the value of x_2 also depends on the value of x_1. In that case a change in x_1 will have two effects on y. First, there will be a direct effect of change in x_1 on y. Second, there will be an indirect effect via x_2 i.e., change in x_1 will affect x_2 and in turn, change in x_2 will affect y. If we like to know the direct effect and ignore the indirect effect, then we have to differentiate y with respect to x_1, assuming x_2 as remaining unchanged. This process of differentiation is called partial differentiation and the result gives us partial derivative. Thus partial derivative in a multivariate function involving two or more independent variables is the derivative with respect to one of the variables, treating all other independent variables as constants. Thus, in the multivariate function $y = f(x_1, x_2, ..., x_n)$ having n independent variables, we have n number of partial derivatives. They are written as $\frac{\delta y}{\delta x_1}, \frac{\delta y}{\delta x_2}, ..., \frac{\delta y}{\delta x_n}$ or more popularly denoted by simple symbols, such as, $f_1, f_2, ..., f_n$, respectively. In a bivariate function $y = f(x_1, x_2)$, we have two partial derivatives, namely, $\frac{\delta y}{\delta x_1}$ and $\frac{\delta y}{\delta x_2}$. Again, they are denoted by f_1 and f_2, respectively. They are called first order partial derivatives with respect to x_1 and x_2, respectively. When we change only one variable, treating others as constants, the multivariate function becomes a function of a single variable. Hence, the same rules of differentiation of a function of single variable are also applicable in the case of partial differentiation.

Example 1.9 : Let us give some examples of partial differentiation.

(i) Let $y = f(x_1, x_2) = 10x_1 + 5x_2$. Obtain f_1 and f_2.

☐ We have, $y = 10x_1 + 5x_2$

$$\therefore \frac{\delta y}{\delta x_1} = f_1 = 10.x_1^{1-1} = 10 \text{ and } \frac{\delta y}{\delta x_2} = f_2 = 5.x_2^{1-1} = 5$$

(ii) Let $y = x_1^3 + 3x_1x_2 + 8x_2^2$. Find f_1 and f_2.

☐ We have $y = x_1^3 + 5x_1x_2 + 8x_2^2$

$$\text{Now, } f_1 = \frac{\delta y}{\delta x_1} = 3.x_1^{3-1} + 5.x_1^{1-1}.x_2 = 3x_1^2 + 5x_2$$

$$\text{Similarly, } f_2 = \frac{\delta y}{\delta x_2} = 5.x_2^{1-1} + 2.8x_2^{2-1} = 5x_1 + 16x_2$$

We should note that while calculating partial derivative with respect to a particular variable, we treat other variables as constant. To signify this the symbol 'δ' is used instead of the notation 'd'. Further, partial derivatives may be themselves functions of the same independent variables as the original function. In our last example, we see that $f_1 = 3x_1^2 + 5x_2 = f_1(x_1, x_2)$ and $f_2 = 5x_1 + 16x_2 = f_2(x_1, x_2)$.

Partial derivatives have important uses in Economics. From the first order partial derivatives, we get marginal values. For example, let the utility function be, $u = f(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are the quantities of n goods, respectively. Here, $\frac{\partial u}{\partial x_1}$ or f_1 is the change in total utility (u) due to one unit change in consumption of x_1. Hence, $\frac{\delta u}{\partial x_1}$ or f_1 is nothing but the marginal utility of x_1. Similarly, $\frac{\delta u}{\delta x_2}$ or f_2 is the marginal utility (MU) of x_2 and so on. In general, $\frac{\partial u}{\partial x_i}$ or f_i is the MU of the i-th commodity (i = 1, 2, ..., n).

Similarly, partial derivative of the production function with respect to a particular input will give us the marginal productivity of that input. For example, let the production function be : $q = f(K,L)$ where q = total product or output, K = amount of capital and L = amount of labour. Now, $\frac{\delta q}{\delta K}$ or f_k is the change in total product due to one unit change in K i.e., marginal productivity of capital. Similarly, $\frac{\delta q}{\delta L}$ or f_L is the marginal productivity of labour.

1.8.2 Total Derivative

Let the bivariate function be : $y = f(x_1, x_2)$. Now suppose that x_1 and x_2 are interdependent. Then, a change in x_1 will have a direct effect on y and an indirect effect through x_2. If we like to consider both the direct effect and indirect effect i.e., the total effect of change in x_1 and x_2 on y, that effect can be known from total derivative. Thus, total derivative of a multivariate function with respect to one independent variable is the sum of both direct effect and indirect effect(s) through other variable(s). For example, let the utility function of the consumer be : $U = f(q_1, q_2)$ where q_1 and q_2 are the quantities of two goods. Then total derivative of U with respect to q_1 is given by : $\frac{dU}{dq_1} = \frac{\delta U}{\partial q_1} + \frac{\partial U}{\partial q_2}.\frac{dq_2}{dq_1}$

Using simpler symbol, $\frac{dU}{dq_1} = f_1 + f_2.\frac{dq_2}{dq_1}$

Similarly, $\frac{dU}{dq_2} = \frac{\partial u}{\partial q_1}.\frac{dq_1}{dq_2} + \frac{\delta U}{\partial q_2} = f_2 + f_1.\frac{dq_1}{dq_2}$

In both expressions, the first term is the direct effect while the second term is the indirect effect.

In this example we have assumed that independent variables are interdependent, i.e., a change in one independent variable affects the other. There may be another type of linkage between the independent variables. Suppose the bivariate function is : $y = f(x_1, x_2)$. Also suppose that both x_1 and x_2 depend on t (time) i.e., $x_1 = g(t)$ and $x_2 = h(t)$. We like to know the rate of change of y due to a change in t i.e., to know $\frac{dy}{dt}$. This $\frac{dy}{dt}$ is called the total derivative of y with respect to t.

Here, change in t does not affect y directly. A change in t affects x_1 and x_2 and changes in x_1 and x_2, in turn, affect y. Now, when t changes, the change in x_1 is $\frac{dx_1}{dt}$. Again, change in x_1 only brings a change in y by $\frac{\delta y}{\delta x_1}$. So, change in y due to change in t is given by $\frac{\delta y}{\delta x_1}.\frac{dx_1}{dt}$. Similarly, change in y due to a small change in t via only x_2 is given by $\frac{\delta y}{\delta x_2} \cdot \frac{dx_2}{dt}$. So the 'total' change in y due to a small change in t (say, dt) is :

$\frac{dy}{dt} = \frac{\delta y}{\delta x_1} \cdot \frac{dx_1}{dt} + \frac{\delta y}{\delta x_2} \cdot \frac{dx_2}{dt}$. Or, using a simple symbol, we have, $\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 . \frac{dx_2}{dt}$.

This $\left(\frac{dy}{dt}\right)$ is our total derivative of y with respect to t. It gives us the rate of change of y due to a change in t.

Example 1.10 : Let us give a simple example. Let $y = f(x_1, x_2) = 3x_1 + 4x_2$.

Further, $x_1 = g(t) = t^2 + t + 1$ and $x_2 = h(t) = t^2 + 3t + 1$. We have to find out $\frac{dy}{dt}$.

Solution : We know that $\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 . \frac{dx_2}{dt}$.

Now, in our example, $f_1 = \frac{\delta y}{\partial x_1} = 3$ and $f_2 = \frac{\partial y}{\delta x_2} = 4$

Again, $x_1 = t^2 + t + 1$. So, $\frac{dx_1}{dt} = 2t + 1$. Further, $x_2 = t^2 + 3t + 1$. So, $\frac{dx_2}{dt} = 2t + 3$.

Putting these values in the expression of $\frac{dy}{dt}$, we get,

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 . \frac{dx_2}{dt} = 3(2t + 1) + 4(2t + 3) = 6t + 3 + 8t + 12 = 14t + 15$$

Check : We may get the same result if we put values of x_1 and x_2 in terms of t in the expression of y and then directly differentiate y with respect to t. Obviously, the result will give us the value of $\frac{dy}{dt}$. We have, $y = 3x_1 + 4x_2 = 3(t^2 + t + 1) + 4(t^2 + 3t + 1)$ or, $y = 3t^2 + 3t + 3 + 4t^2 + 12t + 4 = 7t^2 + 15t + 7$. Thus, y becomes a function of t. Applying power rule of differentiation, we get, $\frac{dy}{dt} = 14t + 15$. We got the same result using the formula of $\frac{dy}{dt}$.

1.8.3 Total Differential of a Multivariate Function

By total differential of a multivariate function we mean the total change in the dependent variable due to change in all the independent variables when independent variables have no interdependence among themselves. Let the multivariate function be : $y = f(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are independent. Now, the rate of change of y due to a small change in x_1, keeping other variables as constant, is $\frac{\delta y}{\partial x_1}$, or in short symbol, f_1. If the amount of change in x_1 is dx_1, then the amount of change in y due to change in x_1 only is given by $f_1.dx_1$. Similarly, if x_2 changes by dx_2, the amount of change in y is given by $f_2.dx_2$, and so on. So, total change in y, say, dy due to change in all n independent variables will be equal to :

$dy = f_1\, dx_1 + f_2dx_2 + ... + f_n\, dx_n$.

dy is called the total differential of the function $y = f(x_1, x_2, ..., x_n)$.

If we take a bivariate function where $y = f(x_1, x_2)$, then if x_1 changes by dx_1 and x_2 changes by dx_2, then the total change in y(denoted by dy) is given by :

$$dy = f_1\, dx_1 + f_2dx_2 = \frac{\partial y}{\partial x_1}.dx_1 + \frac{\partial y}{\partial x_2}.dx_2$$

dy is called the total differential of the function : $y = f(x_1, x_2)$.

Example 1.11 : (i) Find the total differential of the function : $y = 2x_1^2 + 3x_2$

Solution : From the given function, we get, $f_1 = \frac{\delta y}{\partial x_1} = 4x_1$ and $f_2 = \frac{\delta y}{\partial x_2} = 3$.

So, total change in y is : $dy = f_1dx_1 + f_2dx_2 = 4x_1dx_1 + 3dx_2$

(ii) Find the total differential of y when $y = \frac{5}{x_1} + 4x_2^3$

Solution : From the given function, we get, $f_1 = \frac{\delta y}{\partial x_1} = -\frac{5}{x_1^2}$ and $f_2 = \frac{\delta y}{\partial x_2} = 3.4.x_2^{3-1} =$ $12x_2^2$. Then, total differential of y, say, $dy = f_1dx_1 + f_2dx_2$

or, $dy = -\frac{5}{x_1^2}.dx_1 + 12x_2^2.dx_2$

1.8.4 Rules of Total Differential

The rules of total differential are strikingly similar to the rules of derivative of a univariate function.

Rule 1 : If y = k where k is a constant, then dy = dk = 0

Rule 2 : If $y = ku^n$ where u is a function of x_1, then $dy = d(ku^n) = nku^{n-1}.du$

Rule 3 : If y = u ± v where u and v are two functions of x_1 and x_2, respectively, then dy = d(u ± v) = du ± dv

We may generalise this rule. If y = u ± v ± w, then dy = d(u ± v ± w) = du ± dv ± dw.

Rule 4 : If y = u.v where u and v are two functions of x_1 and x_2, respectively, then dy = d(u.v) = v.du + u.dv

We may generalise this rule. Let y = u.v.w. Then dy = d(uvw) = vwdu + uwdv + uvdw

Rule 5 : If $y = \frac{u}{v}$ where u and v are two functions of x_1 and x_2, respectively, then

$$dy = d\left(\frac{u}{v}\right) = \frac{v.du - u.dv}{v^2}$$

1.9 Higher Order Partial Derivatives

Simply speaking, higher order partial derivatives are the derivatives obtained by repitition of partial differentiation. When we repeat the process of partial differentiation, we get the higher order partial derivatives. We know that partial derivatives of a function are generally functions of the same variables of the primary or primitive function. Thus, if $y = f(x_1, x_2)$ then the first order partial derivatives are also generally functions of x_1 and x_2, i.e., $f_1\left(\equiv \frac{\delta y}{\delta x_1}\right) = f_1(x_1, x_2)$ and $f_2\left(\equiv \frac{\delta y}{\delta x_2}\right) = f_2(x_1, x_2)$. In this case, we can repeat the process of partial differentiation and get higher order partial derivatives. This will hold so long partial derivatives are functions of the same variables as in the primitive or primary function. When the partial derivative after some repititions of partial differentiation ceases to be a function of the same variable, further higher order partial derivatives are not obtainable.

Example 1.12 : (i) Let $y = 3x_1^2 + 4x_1.x_2^2$. Obtain higher order partial derivatives.

Solution : Here $f_1 = \frac{\delta y}{\delta x_1} = 6x_1 + 4x_2^2$ and $f_2 = \frac{\delta y}{dx_2} = 8x_1x_2$. Now, second order partial derivative of y with respect to x_1 is : $\frac{\delta^2 y}{dx_1^2} = f_{11} = 6$ and $\frac{\delta^2 y}{dx_2^2} = f_{22} = 8x_1$. Here $f_{11} = 6$ is a constant. So we can further differentiate it only once, i.e., $\frac{\delta^3 y}{dx_1^3} = 0$. It cannot be differentiated further. Similarly, in our example, $\frac{\delta^3 y}{\delta x_2^3} = 0$. So, it cannot also be differentiated further.

(ii) Find f_{11} and f_{22} for the function : $y = (x_1 + 4x_2)^3$

Solution : Here $\frac{\partial y}{\partial x_1} = f_1 = 3(x_1 + 4x_2)^2$ and $\frac{\partial y}{\partial x_2} = f_2 = 3(x_1 + 4x_2)^2.4$.

$$\text{Now, } f_{11} = \frac{\partial}{\partial x_1}(f_1) = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\delta x_1^2} = 3 \times 2\,(x_1 + 4x_2)^1 = 6(x_1 + 4x_2)$$

$$\text{Similarly, } f_{22} = \frac{\delta}{\partial x_2}(f_2) = \frac{\delta}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\delta x_2^2} = 2 \times 12\,(x_1 + 4x_2)^{2-1}.4 = 96(x_1 + 4x_2).$$

We can repeat the process of partial differentiation further. The process will stop at the step when the value of higher order partial derivative becomes zero.

(ii) Let $z = 3x + 5y$. Determine Z_{yy} and Z_{xx}

Solution : Here, $z = 3x + 5y$.

$$\text{So, } z_x = \frac{\partial z}{\partial x} = 3 \text{ and } z_y = \frac{\partial z}{\partial y} = 5$$

$$\text{Now, } \frac{\partial}{\partial x}(z_x) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = 0.$$

$$\text{Similarly, } z_{yy} = \frac{\partial}{\partial y}(z_y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = 0$$

In these examples we have f_{11} or f_{xx} and f_{22} or f_{yy}. These are called direct second order partial derivatives. But there may be other type of partial derivatives. We may want to know the change in f_1 due to change in x_2 or to know the change in f_2 due to change in x_1. In symbols, we may want to know f_{12} or f_{21}. Consider the function $y = f(x_1, x_2)$. Here we have two independent variables : x_1 and x_2. By the process of partial differentiation, we get two first order partial derivatives, say, $\frac{\delta f}{\partial x_1}$ or $\frac{\partial y}{\partial x_1}$ or f_1 and $\frac{\partial f}{\partial x_2}$ or $\frac{\partial y}{\partial x_2}$ or f_2. Now, these first order partial derivatives f_1 and f_2 are generally functions of x_1 and x_2 i.e., $f_1 = f_1(x_1, x_2)$ and $f_2(x_1, x_2)$. So, we shall get four second order partial derivatives. They are :

$$\frac{\partial f_1}{\partial x_1} \equiv \frac{\partial}{\partial x_1}\left(\frac{\partial y_1}{\partial x_1}\right) \equiv \frac{\partial^2 y}{\partial x_1^2} \equiv f_{11}, \quad \frac{\partial f_2}{\partial x_2} \equiv \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) \equiv \frac{\partial^2 y}{\partial x_2^2} \equiv f_{22}$$

$$\frac{\partial f_2}{\partial x_1} \equiv \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) \equiv \frac{\partial^2 y}{\partial x_1 \partial x_2} \equiv f_{12} \text{ and } \frac{\partial f_1}{\partial x_2} \equiv \frac{\partial f_1}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right) \equiv \frac{\partial^2 y}{\partial x_2 \partial x_1} \equiv f_{21}$$

Among these four second order partial derivatives, the first two, f_{11} and f_{22}, are called direct second order partial derivatives and the last two i.e., f_{12} and f_{21}, are called cross second order partial derivatives or mixed partial derivatives. Thus, if the multivariate function has two independent variables, we have four second order partial derivatives (2 direct and 2 cross). If the multivariate function has 3 independent variables, we have $9(= 3^2)$ second order partial derivatives (3 direct and 6 or $3^2 - 3$ cross derivatives) In general, if the multivariate function has n independent variables, it will have n^2 second order partial derivatives (n direct and $(n^2 - n)$ cross or mixed derivatives).

One important result about cross partial derivatives is that if f_1 and f_2 are all continuous and smoth functions of x_1 and x_2, then cross-partial derivatives will be equal i.e., $f_{12} = f_{21}$. This is known as Young's Theorem.

Example 1.13 : (i) Let $y = 4x^3 - 3x_1^2 x_2 + 10x_2^3$. Find f_{11}, f_{22}, f_{12} and f_{21}.

Solution : Here, $f_1 = \frac{\partial y}{\partial x_1} = 12x_1^2 - 6x_1 x_2$ and $f_2 = \frac{\partial y}{\partial x_2} = -3x_1^2 + 30x_2^2$

Now, $f_{11} \equiv \frac{\partial f_1}{\partial x_1} \equiv \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right) \equiv \frac{\partial^2 y}{\delta x_1^2} = 24x_1 - 6x_2$

Similarly, $f_{22} = \frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) \equiv \frac{\partial^2 y}{\partial x_2^2} = 2 \times 30.\, x_2^{2-1} = 60x_2$

Again, $f_{12} = \frac{\partial}{\partial x_1}(f_2) \equiv \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) \equiv \frac{\partial^2 y}{\partial x_1 \delta x_2} = -2.3.\, x_1^{2-1} = -6x_1$

Again, $f_{21} = \frac{\partial}{\partial x_2}(f_1) \equiv \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right) \equiv \frac{\partial^2 y}{\partial x_2 \partial x_1} = -6x_1(1) = -6x_1$

It should be noted that $f_{12} = f_{21} = -6x_1$ i.e., Young's theorem holds.

(ii) Given $z = 5x^3y - 20xy + 8xy^3$. Obtain f_{xx} and f_{yy} and check whether Young's theorem holds (or check whether cross-partial derivatives are equal or not).

Solution : We have, $z = 5x^3y - 20xy + 8xy^3 = f(x, y)$

Here $f_x \equiv \frac{\delta z}{\delta x} = 15x^2y - 20y + 8y^3$ and $f_y \equiv \frac{\partial z}{\partial x} = 5x^3 - 20x + 24xy^2$

Now, $f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 z}{\partial x^2} = 30xy,$

and $f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 z}{\partial y^2} = 2 \times 24xy^{2-1} = 48xy$

Now we consider the values of f_{xy} and f_{yx} i.e., the values of cross-partial derivatives.

$$f_{xy} \equiv \frac{\partial}{\partial x}(f_y) \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \equiv \frac{\partial^2 z}{\partial x \partial y} = 15x^2 - 20 + 24y^2$$

$$f_{yx} \equiv \frac{\partial}{\partial y}(f_x) \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \equiv \frac{\partial^2 z}{\partial y \partial x} = 15x^2 - 20 + 24y^2 = f_{xy}$$

Thus, cross-partial derivatives are equal, or, in other words, Young's theorem holds.

1.10 Homogeneous Function

Mathematically speaking, a function $y = f(x_1, x_2, ... x_n)$ is said to be homogeneous of degree k if $(\lambda x_1, \lambda x_2, ... \lambda x_n) = \lambda^k.y$. Thus, in language, a function is sand to be homogeneous of degree k if multiplication of its each independent variable by a constant λ changes the value of the dependent variable i.e., the value of the function by λ^k times. On the other hand, if λ cannot be factored out, the function is said to be non-homogeneous. The power of λ i.e., k is called the degree of homogeneity. Thus, a bivariate function $y = f(x_1, x_2)$ is said to be homogeneous of degree k of $f(\lambda x_1, \lambda x_2) = \lambda^k.y$. The degree of homogeneity of a homogeneous function can easily be calculated by a simple technique. For a homogeneous function, the sum of indices for each term of the function is the same and the sum of indices will be the degree of homogeneity. Thus, $y = x_1^3 + x_2^3$, $y = 3x_1^2 + 2x_1x_2 + 4x_2^2$, $y = x_1^{\frac{1}{4}}x_2^{\frac{3}{4}}$ are homogeneous functions. The first function is homogeneous of degree 3, the second function is homogeneous of degree two while the last function having only a single term is homogeneous of degree 1. Let us check our statement.

To consider the degree of homogeneity of the function, $y = x_1^3 + x_2^3$, we increase the independent variables by λ times. The new value of y, say, $y^* = (\lambda x_1)^3 + (\lambda x_2)^3 = \lambda^3(x_1^3 + x_2^3) = \lambda^3.y$. Thus, the given function is homogeneous of degree 3. For the second function, $y = 3x_1^2 + 2x_1x_2 + 4x_2^2$, we increase x_1 and x_2 by λ times. The new value of y = y* (say) = $3(\lambda x_1)^2 + 2(\lambda x_1)(\lambda x_2) + 4(\lambda x_2)^2 = \lambda^2(3x_1^2 + 2x_1x_2 + 4x_2^2) = \lambda^2.y$. Thus the given function its homogeneous of degree 2. Let us consider the degree of homogeneity of the third function, $y = x_1^{\frac{1}{4}}x_2^{\frac{3}{4}}$. If we increase both x_1 and x_2 by λ times, the new value of y, say, $y^* = (\lambda x_1)^{\frac{1}{4}}(\lambda x_2)^{\frac{3}{4}} = \lambda^{\frac{1}{4}+\frac{3}{4}}.x_1^{\frac{1}{4}}x_2^{\frac{3}{4}} = \lambda^1.y = \lambda y$. Thus the given function is homogeneous of degree 1. It is also called linearly homogeneous function.

On the other hand, functions like $y = x_1^3 + x_2^4$, $y = x_1^4 + x_1x_2 + x_2^3$, $y = x_1x_2 + x_2 + x_1$ are examples of non-homogeneous functions. Homogeneous functions have many applications in Economics. We shall consider some of them in our next unit.

Example 1.14 : (i) Determine the degree of homogeneity of the function, $z = ax^2 + by^2$.

Solution : We increase both the independent variables x and y by λ times. The new value of z, say, $z^* = a(\lambda x)^2 + b(\lambda y)^2 = \lambda^2(ax^2 + by^2) = \lambda^2.z$. Hence the degree of homogeneity of the given function is 2.

(ii) Let the function be : $q = ak^{\alpha}L^{1-\alpha}$. Determine its degree of homogeneity.

Solution : We increase the values of K and L by λ times. The new value of q = q*(say) $= a(\lambda k)^{\alpha}(\lambda L)^{1-\alpha} = \lambda^{\alpha+1-\alpha}.\ ak^{\alpha}\ L^{1-\alpha} = \lambda^1.q = \lambda q$. Hence the given function is homogeneous of degree 1.

(iii) Suppose the bivariate function is : $y = Ax_1^{\alpha}x_2^{\beta}$. What is its degree of homogeneity?

Solution : We increase both x_1 and x_2 by λ times. the new value of y, say, $y^* = A(\lambda x_1)^{\alpha}(\lambda x_2)^{\beta} = \lambda^{\alpha+\beta}.\ Ax_1^{\alpha}x_2^{\beta} = \lambda^{\alpha+\beta}.\ y$. So, the given function is homogeneous of degree $(\alpha + \beta)$.

(iv) Determine the degree of homogeneity of the function : $z = ax^2 + by^2 + c$.

Solution : Raising x and y by λ times, we get the new value of z, say, $z^* = a(\lambda x)^2 +$

$b(\lambda y) + c = \lambda^2ax^2 + \lambda by + c$. Here, λ cannot be factored out. So the given function is non-homogeneous.

1.11 Euler's Theorem on Homogeneous Function

The Euler's theorem states that if a multivariate function $y = f(x_1, x_2 \ldots, x_n)$ is homogenous of degree k, then $x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\delta x_2} + \ldots + x_n . \frac{\partial y}{\partial x_n} = k.y$

Using alternative notation, the Euler's theorem states that

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\delta f}{\partial x_2} + \ldots + x_n . \frac{\partial f}{\partial x_n} = k.f$$

If we use symbols for partial derivatives, we may write the Euler's theorem as follows :

$x_1f_1 + x_2.f_2 + \ldots + x_nf_n = k.y$

We shall prove this theorem taking a bivariate function involving two independent variables x_1 and x_2. However the result can easily be generalised for n number of independent variables.

Now, in our case, $y = f(x_1, x_2)$. We assume that this function is homogeneous of degree k. So we have to prove that $x_1 . \frac{\partial y}{\partial x_1} + x_2 . \frac{\partial y}{\partial x_2} = k.y$ or, using f for y, we have to prove that $x_1 . \frac{\partial f}{\partial x_1} + x_2 . \frac{\partial f}{\partial x_2} = kf$ or, using the notation of partial derivative, we have to prove that $x_1.f_1 + x_2.f_2 = k.y$.

Proof : Our given function is : $y = f(x_1, x_2)$. This function is assumed to be homogeneous of degree k.

Hence, by definition of homogeneous function, we may write, $\lambda^k.y = f(\lambda x_1, \lambda x_2)$

Putting $\lambda = \frac{1}{x_1}$, we get, $\frac{1}{x_1^k}.y = f\left(1, \frac{x_2}{x_1}\right)$

$$\therefore\ y = x_1^k.\phi\left(\frac{x_2}{x_1}\right) \text{ where } f\left(1, \frac{x_2}{x_1}\right) = \phi\left(\frac{x_2}{x_1}\right) \qquad \ldots (1)$$

Differentiating both sides of equation(1) with respect to x_1, we get,

$$\frac{\partial y}{\partial x_1} \equiv \frac{\partial f}{\partial x_1} = kx_1^{k-1}.\phi\left(\frac{x_2}{x_1}\right) + x_1^k.\phi'\left(\frac{x_2}{x_1}\right)\left(-\frac{x_2}{x_1^2}\right)$$

Multiplying both sides by x_1, we get,

$$x_1.\frac{\partial y}{\partial x_1} \equiv x_1.\frac{\partial f}{\partial x_1} = kx_1^k.\phi\left(\frac{x_2}{x_1}\right) - x_1^k.\phi'\left(\frac{x_2}{x_1}\right).\frac{x_2}{x_1} \qquad \text{... (2)}$$

Again, differentiating (1) with respect to x_2, we get, $\frac{\partial y}{\partial x_2} \equiv \frac{\partial f}{\partial x_2} = x_1^k\phi'\left(\frac{x_2}{x_1}\right).\frac{1}{x_1}$

Multiplying both sides by x_2, we get,

$$x_2.\frac{\partial y}{\partial x_2} \equiv x_2.\frac{\partial f}{\partial x_2} = x_1^k.\phi'\left(\frac{x_2}{x_1}\right).\frac{x_2}{x_1} \qquad \text{... (3)}$$

Adding (2) and (3) we get, $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} \equiv x_1.\frac{\partial f}{\partial x_1} + x_2.\frac{\partial f}{\partial x_2}$

$$= kx_1^k.\phi\left(\frac{x_2}{x_1}\right) = ky \text{ [from(1)]}$$

This proves our theorem. In the similar fashion, we can generalise the theorem for n independent variables.

Example 1.15 : (i) Check whether Euler's theorem holds for the function, $y = 3x_1^2 + 5x_2^2$

Solution : Here the given function $y = 3x_1^2 + 5x_2^2$ is homogeneous of degree 2. If we increase both x_1 and x_2 by λ times, the new value of y, say, $y^* = 3(\lambda x_1)^2 + 5(\lambda x_2)^2 = \lambda^2(3x_1^2 + 5x_2^2) = \lambda^2.y$. So the given function is homogeneous of degree 2. Now, the Euler's theorem states that if $y = f(x_1, x_2)$ is homogeneous of degree n, then $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} = ny$. So, in our context, the Euler's theorem will hold if $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} = 2y$. Let us examine it. Here $\frac{\partial y}{\partial x_1} = 2 \times 3x_1$ and $\frac{\partial y}{\partial x_2} = 2 \times 5x_2$

Now, $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} = 2 \times 3x_1^2 + 2 \times 5x_2^2 = 2(3x_1^2 + 5x_2^2) = 2y$

Thus, in this case, Euler's theorem holds.

(ii) Let $z = x^3 + 3x^2y + 3xy^2 + y^3$. Prove that $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 3z$

Proof : We have, $z = x^3 + 3x^2y + 3xy^2 + y^3 = f(x, y)$

Now, $\frac{\partial z}{\partial x} = 3x^2 + 6xy + 3y^2$ and $\frac{\partial z}{\partial y} = 3x^2 + 6xy + 3y^2$

Now, LHS = $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = (3x^3 + 6x^2y + 3xy^2) + (3x^2y + 6xy^2 + 3y^3)$

$= 3x^3 + 9x^2y + 9xy^2 + 3y^3 = 3(x^3 + 3x^2y + 3xy^2 + y^3) = 3z =$ RHS.

[Note : Here the degree of homegeneity of the given function is 3. Hence, as per Euler's theorem, $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 3z$]

(iii) Let $z = x^{\alpha}y^{1-\alpha}$. Prove that $x.z_x + y.z_y = z$

Proof : we have, $z = x^{\alpha}y^{1-\alpha}$

Now, $z_x = \frac{\partial z}{\partial x} = \alpha\, x^{\alpha-1}\, y^{1-\alpha}$ and $z_y = \frac{\partial z}{\partial y} = (1-\alpha)x^{\alpha}y^{-\alpha}$

Now, LHS $= x.z_x + y.z_y = x(\alpha x^{\alpha-1}\, y^{1-\alpha}) + y(1-\alpha)x^{\alpha}\, y^{-\alpha}$

$= \alpha x^{\alpha}\, y^{1-\alpha} + (1-\alpha)x^{\alpha}\, y^{1-\alpha} = x^{\alpha}\, y^{1-\alpha}\,(\alpha + 1 - \alpha)$

$= x^{\alpha}\, y^{1-\alpha} = z =$ RHS (Proved)

Actually, here the given function is homogeneous of degree 1. Hence the Euler's theorem holds.

(iv) If $y = x_1^{\alpha}x_2^{\beta}$, prove that $x_1.f_1 + x_2.f_2 = (\alpha + \beta)y$.

Solution : We have, $y = x_1^{\alpha}x_2^{\beta}$

Now, $f_1 \equiv \frac{\partial y}{\delta x_1} = \alpha.x_1^{\alpha-1}x_2^{\beta}$ and $f_2 \equiv \frac{\partial y}{\delta x_2} = \beta.x_1^{\alpha}x_2^{\beta-1}$

$\therefore\ x_1.f_1 + x_2.f_2 = \alpha x_1^{\alpha}x_2^{\beta} + \beta x_1^{\alpha}x_2^{\beta} = x_1^{\alpha}x_2^{\beta}(\alpha+\beta) = (\alpha + \beta)y =$ RHS (proved)

[Note : Here the given function, $y = x_1^{\alpha}x_2^{\beta}$ is homogeous of degree $(\alpha + \beta)$. So as per Euler's theorem, $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} = ny = (\alpha + \beta)y$].

1.12 Concept of Homothetic Function

Homothetic function is a generalised class of homogeneous function. Thus, if Q = f(K, L) is a homogeneous function, then z = F(Q) = F[f(K, L)] is homothetic if $\frac{dz}{dQ} > 0$.

In other words, a homothetic function can be derived from a monotonic transformation of a homogeneous function. A homothetic function may not be a homogeneous function. For example, let Q = aK + bL. Here Q is a homogeneous function of degree 1 [If we increase K and L by λ times, the new value of Q, say, Q* = a(λK) + b(λL) = λ(aK + bL) = λ^1.Q = λQ]

New, let z = aK + bL + c where c is a positive constant. Then z = q + c and $\frac{dz}{dq} = 1 > 0$. Hence, z is a homothetic function which is a monotonic transformation of the homogeneous function, Q = aK + bL. However, z = aK + bL + c (c > 0), though homothetic, is not a homogeneous function.

1.13 Summary

1. DEFINITION AND TYPES OF FUNCTIONS

Functions are mathematical expressions showing dependency between two variables or among more than two variables. Functions may be of different types, such as constant function, linear function, quadratic function, cubic function, or, in general, polynomial function of degree n.

2. CONCEPTS OF DERIVATIVE AND DIFFERENTIATION

The concept of derivative gives us the rate of change of the dependent variable when the independent variable(s) changes (change) infinitesimally. The act or technique of finding out the value of derivative is called differentiation. These are some standard rules of differentiation or rules for finding out derivative of a function.

3. CONCEPT OF HIGHER ORDER DERIVATIVES OR HIGHER ORDER DIFFERENTIATION

When a function is differentiated for the first time, the resultant derivative is called the first order derivative. If we repeat the process of differentiation, we shall get higher order derivatives. Thus, the second order derivative is the derivative of the first order derivative; a third order derivative is the derivative of the second order dervative, and so on. The process of differentiation will reach the final stage when the derivative becomes a constant function. Its further differentiation will give the value zero and the process of higher order differentiation stops. If y = f(x) then its first order derivative, denoted by $\frac{dy}{dx}$, gives us the rate of change of the dependent variable(y) due a very small change in the independent variable (x). The second order derivative of y, denoted by $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ gives us the rate of change in $\frac{dy}{dx}$ due to a very small change is x. Similarly we can interprete $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ and so on.

4. SLOPE AND CURVATURE

Slope of a linear function is the tan of the angle between the line and the horizontal(x) axis on its positive direction. Slope of a non-linear function at any point on it is the tan of the angle between the tangent at that point and the horizontal axis on its positive direction. If y = f(x), then the slope of the function is measured by its first derivative i.e., $\frac{dy}{dx}$.

The curvature of a function can be known from the sign of the second derivative $\left(\frac{d^2y}{dx^2}\right)$ i.e., change in the value of $\frac{dy}{dx}$ or of the first derivative.

5. MULTIVARIATE FUNCTIONS AND THEIR DERIVATIVES

A function having more than one independent variable is called a multivariate function. A special case of multivariate function is the bivariate function which has two independent variables. When a function has one independent variable, it may be called univariate function. By function we simply or generally mean this univariate function if not otherwise mentioned.

In the case of multivariate function, we have three types of derivatives, namely, partial derivative, total derivative and total differential. The rules of finding out these derivatives of a multivariate function are very much similar to those of finding out $\frac{dy}{dx}$ in the case of a univariate function.

6. HIGHER ORDER PARTIAL DERIVATIVES

Higher order partial derivatives are simply the derivatives obtained by repitition of partial differentiation. In the case of univariate function, we get higher order derivatives just by repeating the process of differentiation. Similarly, in the case of a multivariate function, if we repeat the process of partial differentiation, we get the higher order partial derivatives.

7. HOMOGENEOUS FUNCTION

A function is said to be homogeneous of degree k if multiplication of its each independent variable by a constant λ will change the value of the function by the proportion λ^k. In symbols, the multivariate function $y = f(x_1, x_2, ..., x_n)$ is said to be homogeneous of degree k if $f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^k.y$.

8. EULER'S THEOREM

Euler's theorem states that if a function is homogeneous of degree k, then the sum of its all partial derivatives multiplied by the corresponding variable will be equal to the function multiplied by k. In symbols, if $y = f(x_1, x_2, ..., x_n)$ is homogeneous of degree k,

then Euler's theorem states that $x_1.\frac{\partial f}{\partial x_1} + x_2.\frac{\partial f}{\partial x_2} + ... + x_n.\frac{\partial f}{\partial x_n} = k.f$

Or, using alternative notation, $x_1.\frac{\partial y}{\partial x_1} + x_2.\frac{\partial y}{\partial x_2} + ... + x_n.\frac{\partial y}{\partial x_n} = k.y$.

This theorem has important applications in various economic concepts.

9. HOMOTHETIC FUNCTION

A homothetic function is a generalisation of homogeneous function.

1.14 Key Concepts

1. Function : Two variables are said to be functionally related if for a particular value of one variable we get a particular value of the other.

2. Dependent variable : The variable whose value is dependent or determined by the value(s) of independent variable(s) is known as dependent variable

3. Independent variable : The variable whose value is determined independently of or outside the system, is called independent variable.

4. Variable : Variable means anything whose value varies or changes.

5. Polynomial equation : Polynomial equation is an equation by which, in general, several terms in an independent variable are raised to various powers. The degree of the polynomial is the highest power to which the independent variable is raised.

6. Linear Function : Linear function is a mathematical relationship in which the variables appear as additive elements, with no multiplicative or exponential components. The general form of a linear function is : $a_0 + a_1x_1 + a_2x_2 + ... + a_nx_n = 0$.

7. Quadratic equation : Quadratic equation is an equation which involves the square of a variable as the highest power. The general form of a quadratic equation is : $y = ax^2 + bx + c$ where a, b and c are constants.

8. Cubic equation : A cubic equation is an equation in which the highest power of an independent variable is three (i.e., its cube). For example, $y = a_0 + a_1x + a_2x^2 + a_3x^3$ is a cubic equation (provided $a_3 \neq 0$)

9. Rational function : A function expressed as a ratio of two polynomial functions, is known as rational function.

10. Rectangular hyperbola : Rectangular hyperbola is such a curve that the area of all the rectangles obtained by joining abscissa and ordintate of all points on this curve is constant. The equation of a rectangular hyperbola is : $xy = k$ where k is a constant.

11. Algebraic function : Any function expressed in terms of polynomials and/or roots, such as, square root of polynomials is an algebraic function.

12. Derivative : The change in the dependent variable of a function per unit change in independent variable, calculated for an infinitesimally small interval for the latter, is known as derivative of the function.

13. Differentiation : The process of calculating the derivative of a function is called differentiation.

14. Inverse function : A function whose dependent and independent variables of the original function are interchanged, is called an inverse function.

15. Slope of a linear function : The slope of a linear function is the tan of the angle between the line and the horizontal axis on its positive direction.

16. Slope of a non-linear function : The slope of a non-linear function at any point on it is the tan of the angle between the tangent at that point and the horizontal axis on its positive direction.

17. Multivariave function : A function having more than one independent variable is called a multivariate function.

18. Bivariate function : A special multivariate function whose number of independent variables is just two, is known as bivariate function.

19. Partial derivative : Partial derivative in a multivariate function involving two or more independent variables is the derivative with respect to one of the variables, treating all other independent variables as constants.

20. Total derivative : Total derivative of a multivariate function with respect to one independent variable is the sum of both direct effect and indirect effect(s) through other variable(s).

21. Total differential : Total differential of a multivariate function is the total change in the dependent variable due to change in all the independent variables when independent variables have no interdependence among themselves.

22. Higher order partial derivatives : Higher order partial derivatives are the derivatives obtained by repitition of partial differentiation.

23. Homogeneous function : A function $y = f(x_1, x_2, ..., x_n)$ is said to be homogeneous of degree k if $f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^k.y$.

24. Euler's theorem : The Euler's theorem states that if a multivariate function $y = f(x_1, x_2, ..., x_n)$ is homogeneous of degree k, then $= k.y$.

25. Homothetic function : A homothetic function is a monotomically increasing function of any homogeneous function.

1.15 Exercises

A. Short Answer Type Questions

1. Define function.
2. What is constant function?
3. Give the definition of polynomial function.
4. What is inverse function?
5. What do you mean by rational function?
6. What is a rectangular hyperbola?
7. Define non-algebraic function.
8. What are the different types of non-algebraic function?
9. What is derivative of a function?
10. What is differentiation?
11. State the first principle of differentiation.
12. State the power rule of differentiation.

13. State the product rule of differentiation.

14. Let $y = e^{7x}$. Determine $\frac{dy}{dx}$.

15. Draw a constant function on (x,y) plane.

16. What is partial derivative of a function?

17. What do you mean by total derivative of a function?

18. What is total differential of a function?

19. Define a multivariate function.

20. What is a bivariate function?

21. What do you mean by higher order partial derivatives?

22. What is Young's theorem?

23. What is homogeneous function?

24. Determine the degree of homogeneity in the following cases :

(i) $z = \frac{x_1^2}{x_2}$ (ii) $y = a^{1-b}c^b$ (iii) $z = \frac{x}{y}$ (iv) $z = \frac{x^5 + y^5}{x^2 + y^2}$

26. State the Euler's theorem.

Medium Answer Type Questions (Each of 5 marks)

1. Write a short note on the concept of function.
2. Explain the concept of inverse function with a suitable example.
3. Briefly describe the concept of rational function.
4. Determine $\frac{dy}{dx}$ of the following function from first principle of differentiation : $y = 7x^2 - 8x + 20$
5. Explain the chain rule or function of a function rule of differentiation.
6. State the quotient rule of differentiation. Give on example to clarify the rule.
7. Write a short note on higher order derivatives or higher order differentiation.
8. How can you determine slope of a linear function?
9. How will you determine slope of a non-linear function?
10. Briefly describe the concept of partial derivative.
11. Explain the concept of total derivative of a bivariate function.
12. Discuss the concept of total differential of a multivariate function.

13. Mention the major rules of total differential of a function.
14. What do you mean by direct second order partial derivatives and cross second order partial derivatives.
15. State Young's theorem. Show with the help of an example that cross partial dervatives are equal.
16. Define homogeneous function and show how the degree of homogeneity can be determined.
17. Determine the degree of homogeneity of the following two functions :

(a) $y = 30x_1^{\alpha}x_2^{\beta}$ (b) $z = A[\alpha x^{-\rho} + (1-\alpha)y^{-\rho}]^{-\frac{1}{\rho}}$

Ans. (a) $\alpha + \beta$, (b) 1

Long Answer Type Questions

1. Briefly describe some major types of functions.
2. Discuss different types of polynomial functions.
3. Analyse the concept of derivative or differentiation citing some examples.
4. State the major rules of differentiation or derivative.
5. Briefly describe the concept of slope of a function using suitable diagrams wherever necessary.
6. Discuss the concept of curvature of a function.
7. Write a short note on multivariate functions and their derivatives.
8. Make a clear distinction among partial derivative, total derivative and total differential of a multivariate function.
9. Write a short note on higher order partial derivatives.
10. State the Euler's theorem. Prove the Euler's theorem taking a bivariate function.

1.16 References

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Unit 2 ❒ Applications of Functions and Derivatives in Economics

Structure

2.1 Objectives

After the study of this unit, the reader will be able to know

- application of various functions in Economics
- average and marginal functions
- conditions of profit maximisation/cost minimisation
- slope and curvature of various curves used in Economics
- relation among different concepts of revenue and price elasticity of demand
- properties of homogeneous production function
- properties of Cobb-Douglas production function
- the product exhaustion theorem
- properties of CES production function

2.2 Introduction

In the previous unit, we have learnt about various types of functions and their derivatives. In the present unit, we shall learn about the economic applications of those concepts. In Economics, we come across numerous types of relationships among variables i.e., functions. For example, we have the demand function, D = f(p), where D = quantity demanded and p = price; we may have the supply function, S = S(P), the consumption function, C = f(Y) where Y stands for income, the saving function, S = S(Y) where S = amount of saving, the investment function, I = I(r) where r stands for the rate of interest and so on. These are examples of univariate functions where the number of independent variable is just one. Similarly, we may have functions of more general type like, D = f(P, Y, P_r, t...) where D = demand, P = own price of the good, Y = income of the consumer, P_r = Prices of related goods, t = tastes of the consumer, etc. This is actually a multivariate function. When we write, C = C(Y, r, a, d, ...) where Y = level of income, a = asset holding, r = rate of interest, d = distribution of income, it is also an example of multivariate function. When we say that saving(S) and investment (I) depend on the level of income (Y) and the rate of interest(r), then S = f(Y, r) and I = g(Y, r). They are actually examples of bivariate functions when the number of independent variables is two. Bivariate functions, we know, are special cases of multivariate functions. In Economics, we have many other functions stating some relationships among different variables .

Now, in Economics, most of the economic decisions are determined by the application of the concept of 'marginal'. While taking a decision, the decision-maker (household, firm, government or anybody else) has to consider the marginal (or extra) benefit from and marginal cost of implementing that decision. If the marginal benefit exceeds marginal cost, that decision is undertaken. Now, the magnitudes of marginal benefit and marginal cost of that decision can be known by applying the concept of derivative or the technique of differentiation. Hence, derivative or differentiation plays a very important role in Mathematical Economics. In this unit, we shall try to learn about the applications of functions and their derivatives in Economics.

2.3 Average and Marginal Functions

Suppose we have a function : y = f(x). Then y/x or f(x)/x is the average function of the original function y = f(x). On the other hand, $\frac{dy}{dx}$ or $f'(x)$, i.e., the first derivative of the function is called its marginal function. Consider the example from Economics. Let total revenue (R) is a function of the amount of output sold (q), i.e., R = R(q). So, $\frac{R}{q}$ or, $\frac{R(q)}{q}$ is called the average revenue function. The marginal function is given by $\frac{dR}{dq}$ or $R'(q)$. This $\frac{dR}{dq}$ is the mathematical notation of MR which is the change in total revenue due to one unit change in output. We take another example. Let the consumption function or the propensity to consume be given by : C = C(Y). Here C = total consumption expenditure and Y = total income. So, average function of the consumption function is $\frac{C}{Y}$ or $\frac{C(Y)}{Y}$. In Economics, it is called the average propensity to consume(APC). Again, marginal function of this consumption function is $\frac{dC}{dY}$ or $C'(Y)$. In Economics, it is called the marginal propensity to consume (MPC). Consider another example. Let total cost (C) of a firm depend on the level of output produced (q). So, C = C(q). Then its average function is $\frac{C}{q}$ or $\frac{C(q)}{q}$ while $\frac{dC}{dq}$ or $C'(q)$ is the marginal function. In Economics, the former is called average cost (AC) or average total cost (ATC) while the latter is called marginal cost (MC).

2.4 Different Elasticities in Terms of Average and Marginal Functions

Let the univariate function be : y = f(x). It simply says that y will change as x does.

Suppose we want to know the percentage change in y due to one percent change in x. That is known from the concept of elasticity. Thus, elasticity of y with respect to x(or x-elasticity of y) denoted by e_y is given by :

$$e_y = \frac{\text{percentage change in y}}{\text{percentage change in x}} = \frac{\frac{dy}{y}\times 100}{\frac{dx}{x}\times 100} = \frac{dy}{y}\times\frac{x}{dx} = \frac{x}{y}.\frac{dy}{dx}$$

This may be re-written as, $e_x = \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}}$

Thus, by using the concepts of average function and marginal function we can get the value of elasticity. Take examples from Economics. If the demand function is : D = f(p), then price elasticity of demand,

$$e_d = \frac{\frac{dD}{D}\times 100}{\frac{dP}{P}\times 100} = \frac{dD}{D}\times\frac{P}{dP} = \frac{P}{D}.\frac{dD}{dP} = \frac{\frac{dD}{dP}}{\frac{D}{P}} = \frac{\text{marginal function}}{\text{average function}}$$

Similarly, we have the supply function S = S(P). The elasticity of supply

$$= e_s = \frac{\text{marginal function}}{\text{average function}} = \frac{\frac{dS}{dP}}{\frac{S}{P}} = \frac{P}{S}.\frac{dS}{dP} = \frac{\frac{dS}{S}\times 100}{\frac{dP}{P}\times 100}$$

$$\therefore\ e_s = \frac{\text{percentage change in supply}}{\text{percentage change in price}}$$

If the cost function is c = f(q) where c = total cost and q = total output, then elasticity of total cost with respect to output,

$$e_c = \frac{\frac{dc}{c}}{\frac{dq}{q}} = \frac{q}{c}.\frac{dc}{dq} = \frac{\frac{dc}{dq}}{\frac{c}{q}} = \frac{\text{marginal function}}{\text{average function}}\text{. Thus, } e_c = \frac{MC}{AC}.$$

For the consumption function C = C(Y) where C = total consumption, Y = total income, elasticity of consumption expenditure with respect to income is,

$$e_C = \frac{\text{percentage change in consumption}}{\text{percentage change in income}} = \frac{\frac{dC}{C}\times 100}{\frac{dY}{Y}\times 100}$$

$$= \frac{dC}{C}\times\frac{Y}{dY} = \frac{Y}{C}.\frac{dC}{dY} = \frac{\frac{dC}{dY}}{\frac{C}{Y}} = \frac{\text{marginal function}}{\text{average function}}. \text{ Thus, } e_C = \frac{MPC}{APC}$$

Thus we can express various elasticities in terms of marginal function and average function.

2.5 Major Applications of Derivatives in Economics

We know that if y = f(x), then its first derivative is given by : $\frac{dy}{dx}$ or $f'(x)$. This derivative has so many applications in Economics. We consider some of them below.

2.5.1 To Determine Different Types of Elasticities

We first consider different types of elasticity of demand. Let us take a multivariate demand function : $q_x = f(p_x, M, p_y)$ i.e., demand for any comodity, x, depends on its own price (p_x), income of the consumer (M) and price of the related good (p_y). So, here we have 3 types of elasticity of demand, namely, (own) price elasticity of demand, income elasticity of demand and cross (price) elasticity of demand.

We first consider (own) price elasticity of demand. In that case we take M and p_y as given. So, $q_x = f(P_x)$ or, simply, q = f(p). We know that price elasticity of demand,

$$e_d = \frac{\text{marginal function}}{\text{average function}} = \frac{\frac{dq}{dp}}{\frac{q}{p}} = \frac{p}{q}.\frac{dq}{dp}$$

From this formula, we can easily determine the value of e_d if the demand function is given. We give some examples. We should note that to determine e_d, we have to know the first derivative of the demand function, i.e., $\frac{dq}{dp}$.

Example 2.1. The demand function is $q = \frac{60}{2p+5}$. Calculate price elasticity of demand at p = 5.

Solution : If p = 5, $q = \frac{60}{2p+5} = \frac{60}{2\times5+5} = \frac{60}{15} = 4$

Again, $\frac{dq}{dp} = -\frac{60}{(2p+5)^2}\times2$

Putting p = 5, $\frac{dq}{dp} = -\frac{60}{(15)^2}\times2 = -\frac{60}{15\times15}\times2 = -\frac{8}{15}$

So, $e_d = \frac{p}{q}\times\frac{dq}{dp} = \frac{5}{4}\times-\frac{8}{15} = -\frac{2}{3}$

Thus, at p = 5, $e_d = -\frac{2}{3}$, or absolute value of $e_d = |e_d| = \frac{2}{3}$

Example 2.2 The demand function is : $D = 74 - 2p - p^2$. Calculate e_d when p = 4.

Solution : When p = 4, $D = 74 - 2p - p^2 = 74 - 2 \times 4 - 4^2 = 50$

Again, $\frac{dD}{dp} = -2 - 2p$. When p = 4, $\frac{dD}{dp} = -2 - 2 \times 4 = -10$

Putting these values, we get,

$e_d = \frac{p}{D}.\frac{dD}{dp} = \frac{4}{50}\times-10 = -\frac{4}{5}$ $\therefore |e_d| = \frac{4}{5} = 0.8$

Example 2.3 : Calculate price elasticity of demand for the function : $x = \frac{100}{p^5}$

Solution : Our demand function is : $x = \frac{100}{p^5} = 100p^{-5}$

$\therefore \frac{dx}{dp} = -5 \times 100p^{-5-1} = -5 \times 100p^{-6}$

Now, $e_d = \frac{p}{x}.\frac{dx}{dp} = \frac{p}{100p^{-5}}\times(-5).100.p^{-6} = (-5).\frac{p^{-5}}{p^{-5}} = -5$

So, $e_d = -5$ or, $|e_d| = 5$.

In this case, the value of $e_d = -5$ for any value of price or quantity demanded. Such demand curves are called iso-elastic demand curves. We consider a general example.

Example 2.4 : Let $q = \frac{a}{p^\alpha}$ or, $q = ap^{-\alpha}$. Calculate e_d of this demand function.

Solution : We have, $q = \dfrac{a}{p\alpha} = ap^{-\alpha}$.

Now, $\dfrac{dq}{dp} = -\alpha.ap^{-\alpha-1}$

We know, $e_d = \dfrac{p}{q}.\dfrac{dq}{dp} = \dfrac{p}{ap^{-\alpha}}(-\alpha)ap^{-\alpha-1} = \dfrac{(-\alpha)p^{-\alpha}}{p^{-\alpha}} = -\alpha$ or, $|e_d| = \alpha$.

Thus, in general, if $q = ap^{-\alpha}$, $e_d = -\alpha$, or, $|e_d| = \alpha$

That is, the value of e_d is the same at all points on the demand curve. We have said that such demand curves are called iso-elastic demand curves. Generally, the exponential demand curves are of this nature.

Alternative method : We can calculate e_d in the case demand functions of exponential nature in an alternative manner. We know that $e_d = \dfrac{\frac{dq}{q}\times 100}{\frac{dp}{p}\times 100} = \dfrac{\frac{dq}{q}}{\frac{dp}{p}} = \dfrac{d\log q}{d\log p}$. This is the definition of e_d in terms of logarithms. From this formula we can easily determine e_d in the case of demand functions of exponential nature. Consider the following example.

Example 2.5 Calculate e_d if the demand law is : $q = ap^{-\alpha}$ $(a > 0, \alpha > 0)$

Solution : We have, $q = ap^{-\alpha}$

Taking log of both sides, we get, $\log q = \log a - \alpha\log p$

Now, we know, $e_d = \dfrac{\frac{dq}{q}}{\frac{dp}{p}} = \dfrac{d\log q}{d\log p}$. This is the formula of price elasticity of demand in terms of logarithms.

$\therefore\ e_d = 0 - \alpha \times 1 = -\alpha$ or, $|e_d| = \alpha$

Thus, if $q = \dfrac{a}{p^{\alpha}}$ or, $q = ap^{-\alpha}$, $e_d = -\alpha$ or, $|e_d| = \alpha$ at all points on the demand curve. One special case of this type of demand function is : $q = \dfrac{a}{p} = ap^{-1}$. In this case, $\alpha = 1$. So, in this case the value of price elasticity of demand $= e_d = -1$. or $|e_d| = 1$. Let us check it.

Example 2.6 : If $q = \frac{a}{p}$, calculate its price elasticity of demand.

Solution : $q = \frac{a}{p} = ap^{-1}$ $\quad \therefore \frac{dq}{dp} = ap^{-2}$

Now, $e_d = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{ap^{-1}} \times -ap^{-2} = -\frac{p^{-1}}{q^{-1}} = -1$

or, $|e_d| = 1$. Here the value of price elastically of demand is unity at all points. Such demand curves are called unit-elastic demand curve. In this case, $q = \frac{a}{p}$

or, pq = a = constant. This is an equation of a rectangular hyperbola.

So, the demand curve will be a rectangular hyperbola in this case. Such curves are also called constant outlay curve or constant expenditure curve because in this case, the expenditure or outlay of the buyer (= pq) is constant (= a). We have considered it in unit 1.

We can calculate e_d in this case by using log-definition also.

Alternative method : Calculate e_d if $q = \frac{a}{p}$ where a = constant(a > 0).

Solution : We have, $q = \frac{a}{p} = ap^{-1}$

Taking log of both sides, we have, log q = log a – log p

Now, $e_d = \frac{d \log q}{d \log p} = 0 - 1 = -1$ $\quad$ or, $|e_d| = 1$

We have seen that if $q = \frac{a}{p^{\alpha}}$ or, $q = ap^{-\alpha}$, then the value of power of p is the value of price elasticity of demand. Hence, if $p = ax^{-n}$, the value of price elasticity of demand will be $\left(-\frac{1}{n}\right)$. Consider the next example.

Example 2.7 : Calculate price elasticity of demand if $p = 20x^{-2}$

Solution : Here, $p = 20x^{-2}$

$$\therefore \frac{dp}{dx} = (-2)\, 20x^{-2-1} = (-2)20x^{-3} = -\frac{2 \times 20}{x^3} \qquad \therefore \frac{dx}{dp} = -\frac{x^3}{2 \times 20}$$

Now, $e_p = \frac{p}{x}.\frac{dx}{dp} = \frac{20x^{-2}}{x} \times \frac{-x^3}{2 \times 20} = -\frac{1}{2}.\frac{x}{x} = -\frac{1}{2}$

Alternative method : We have, p = $20x^{-2}$. Taking log of both sides, we have, log p = log 20 – 2log x

$$\therefore \frac{d \log p}{d \log x} = 0 - 2 \times 1 = -2$$

Now, $e_d = \frac{d \log x}{d \log p}$ $\therefore$ $e_d = \frac{d \log x}{d \log p} = -\frac{1}{2}$ or, $|e_d| = \frac{1}{2} = 0.5$

We may consider the general case. Take the following example.

Example 2.8 : Calculate price elasticity of demand if $p = ax^{-n}$ (a > 0, n > 0).

Solution : We have, $p = ax^{-n}$

Taking log of both sides, log p = log a – n log x

Now, $\frac{d \log p}{d \log x} = 0 - n \times 1 = -n$

$\therefore e_d = \frac{d \log x}{d \log p} = -\frac{1}{n}$, or, $|e_d| = \frac{1}{n}$

Let us consider the calculation of income elasticity of demand. If own price and prices of all related goods remain unchanged, then we can say that quantity demanded (q) of a good will depend on the money income (M) of the consumer, i.e., q = f(M). This is called income-demand function or Engel function. Its graphical form gives us the income-demand curve or the Engel curve. Now, income elasticity of demand may precisely be defined as the percentage change in quantity demanded due to one per cent change in income of the buyer, *ceteris paribus* Thus, income elasticity,

$$e_M = \frac{\text{percentage change in demand}}{\text{percentage change in income}}$$

Using symbols, $e_M = \frac{\frac{dq}{q} \times 100}{\frac{dM}{M} \times 100} = \frac{dq}{q} \times \frac{M}{dM} = \frac{M}{q}.\frac{dq}{dM}$

For normal goods, $\frac{dq}{dM} > 0$ and so, $e_M > 0$.

For inferior goods, $\frac{dq}{dM} < 0$ and so, $e_M < 0$.

We should note that to determine e_M, we should know the first derivative $\left(\frac{dq}{dM}\right)$ of the income demand function or Engel function : q = f(M).

Thus we can say that to determine any sort of elasticity we have to determine the first derivative of the relevant function with respect to the related variable. We consider some examples of determination of income elasticity of demand.

Example 2.9 : Calculate income elasticity of demand if the Engel function is :

q = cM where c is a positive constant.

Solution : We have, q = cM $\therefore \frac{dq}{dM} = c$

Now, $e_M = \frac{dq}{dM}\frac{M}{q} = \frac{cM}{q} = \frac{q}{q} = 1$

Alternative method : We can calculate e_M in this case by using logarithms. In terms of log, the income elasticity of demand may be written as

$$e_M = \frac{\frac{dq}{q}}{\frac{dM}{M}} = \frac{d\log q}{d\log M}$$

[In general, if y = f(x), then x-elasticity of y, say, $e_y = \frac{d\log y}{d\log x}$]

Now, we have, q = cM. So, taking log of both sides, we have, log q = log c + log M

Now, $e_M = \frac{d\log q}{d\log M} = 0 + 1 = 1$

Thus, if the income-demand function or the Engel function is of the form q = cM (i.e. of the standard form : y = mx) or Engel curve is a straight line passing through the origin, the value of income elasticity of demand will be equal to unity in all cases. Thus, if q = 3M or, q = 0.5 M or q = M, the value of income elasticity of demand will be equal to unity in all such cases.

Let us consider the calculation of income elasticity of demand when the demand function is of exponential type. Consider the following example.

Example 2.10 : Calculate price elasticityof demand and income elasticity of demand of the demand function is : $q = Ap^{\alpha} M^{\beta}$. (A, α, β are constants)

Solution : We have, $q = Ap^{\alpha} M^{\beta}$

Taking log of both sides, we have, $\log q = \log A + \alpha \log p + \beta \log M$

Now, price elasticity of demand, $e_P = \dfrac{\partial \log q}{\partial \log p} = 0 + \alpha.1 + 0 = \alpha$

Income elasticity of demand, $e_M \quad \dfrac{\log q}{\log M} = 0 + 0 + \beta.1 = \beta$

If the absolute value of e_P is greater than one, then demand is said to be price-elastic. In the opposite case, demaned is said to be price-inelastic i.e., inelastic with respect to price.

Similarly,if the value of income-elasticity of demand (β) is positive and greater than one, the demand is said to be income elastic. If $\beta > 0$ but $\beta < 1$, demand is said to be income-inelastic. i.e., inelastic with respect to income. If $\beta < 0$, i.e., income elasticity is negative, the good is an inferior good. If $\beta > 1$, the good is called a luxury good. If $0 < \beta < 1$, the good is a necessity.

Let us consider the calculation of cross elasticity of demand. Suppose the demand function is : $x = f(p_x, M, p_y)$ where p_x is the price of good x i.e., own price, M = income of the consumer and p_y is the price of the good y which is somehow related to good x. Now, cross (price) elasticity of demand for good x is given by :

Cross price elasticity, $e_{xy} = \dfrac{\text{percentage change in demand for good x}}{\text{percentage change in price of good y}}$.

Using symbols, $e_{xy} = \dfrac{\frac{\partial x}{x} \times 100}{\frac{\partial p_y}{p_y} \times 100} = \dfrac{\partial x}{x} \times \dfrac{p_y}{dp_y} = \dfrac{p_y}{x} . \dfrac{\partial x}{\partial p_y}$.

We see that to determine cross price elasticity of demand for good x, we have to determine the first derivative of the demand function $x = f(p_x, M, p_y)$ with respect to p_y i.e., we have to know $\dfrac{\partial x}{\partial p_y}$. If x and y are substitutes, the $\dfrac{\partial x}{\partial p_y} > 0$ and so, $e_{xy} > 0$. If x and y are complementary goods, $\dfrac{\delta x}{\partial p_y} < 0$ and so, $e_{xy} < 0$. If x and y are unrelated goods,

$\frac{\partial x}{\partial p_y}$ = 0 and hence e_{xy} = 0. If we use the log-definition, then $e_{xy} = \frac{\partial \log x}{\partial \log p_y}$.

Let us give some examples.

Example 2.11 : Calculate cross (price) elasticity of demand for good x when the demand function is : x = $Ap_x^{\alpha}M^{\beta}p_y^{\gamma}$ where the symbols have their usual meanings (A > 0).

Solution : We have, x = $Ap_x^{\alpha}M^{\beta}p_y^{\gamma}$.

Taking log of both sides, log x = log A + α log p_x + β log M + γ log p_y.

Now, cross price elasticity of demand for good x,

$$e_{xy} = \frac{\partial \log x}{\partial \log p_y} = 0 + 0 + \gamma.1 = \gamma.$$

If γ > 0, x and y are substitutes. If γ < 0, x and y are complements.

Similarly, own price elasticity of demand for good x,

$$e_{d_x} = \frac{\partial \log x}{\partial \log p_x} = 0 + \alpha.1 + 0 + 0 = \alpha$$

Income elasticity of demand for x, $e_{M_x} = \frac{\partial \log\ x}{\partial \log\ M} = 0 + \beta.1 + 0 + 0 = \beta$

Alternative Method : We have, x = $Ap_x^{\alpha}.M^{\beta}p_y^{\gamma}$

$$\therefore\ \frac{\partial x}{\partial p_y} = \gamma A p_x^{\alpha} M^{\beta}.p_y^{\gamma-1} = \frac{\gamma.Ap_x^{\alpha}M^{\beta}.p_y^{\gamma}}{p_y} = \gamma.\frac{x}{p_y}.$$

Now, cross price elasticity of demand, $e_{xy} = \frac{p_y}{x}.\frac{\partial x}{\partial p_y}$. Putting the value of $\frac{\partial x}{\partial p_y}$,

$$e_{xy} = \frac{p_y}{x}.\gamma.\frac{x}{p_y} = \gamma$$

Similarly, (own) price elasticity of demand = α and income elasticity of demand = β

Example 2.12 : Demand functions of two goods are : $q_1 = p_1^{-1.5}p_2^{0.3}$ and $q_2 = p_1^{0.5}p_2^{0.6}$. Calculate cross elasticity of demand for two goods and show the relation between them.

Solution : Demand function of the first good is : $q_1 = p_1^{-1.5}p_2^{0.3}$

∴ log q_1 = –1.5 log p_1 + 0.3 log p_2

$\therefore$ Cross elasticity of demand for q_1, $e_{12} = \dfrac{\delta \log q_1}{\partial \log p_2}$ = 0 + 0.3.1 = 0.3 > 0

Demand function of the second good is : $q_2 = p_1^{0.5} p_2^{-0.6}$

$\therefore$ log q_2 = 0.5 log p_1 – 0.6 log p_2

$\therefore$ Cross elasticity of demand for q_2, $e_{21} = \dfrac{\partial \log q_2}{\partial \log p_1}$ = 0.5 – 0 = 0.5 > 0

As the cross price elasticities of the two goods are positive, the two goods are substitutes of each other.

Let us consider some more examples on price and income elasticities of demand.

Example 2.13 : A consumer's demand curve is : $p = 100 - \sqrt{q}$. Calculate price elasticity of demand if q = 1600.

Solution : When q = 1600, $p = 100 - \sqrt{1600}$ = 100 – 40 = 60

Further, we have, $p = 100 - \sqrt{q} = 100 - q^{\frac{1}{2}}$

$$\therefore \frac{dp}{dq} = -\frac{1}{2}.q^{\frac{1}{2}-1} = -\frac{1}{2}.q^{-\frac{1}{2}} = -\frac{1}{2\sqrt{q}} \qquad \therefore \frac{dq}{dp} = -2\sqrt{q} = -2\sqrt{1600} = -2 \times 40$$

Now, $e_d = \dfrac{p}{q}.\dfrac{dq}{dp} = \dfrac{60}{1600}(-2 \times 40) = -3$, or $|e_d| = 3$

Example 2.14 : If the demand law is $p = (4 - 5x)^2$, for what value of x is the elasticity of demand unity?

Solution : We have, $p = (4 - 5x)^2$

$$\therefore \frac{dp}{dx} = 2(4-5x)(-5) = -10(4-5x) \qquad \therefore \frac{dx}{dp} = -\frac{1}{10(4-5x)}$$

Now, price elasticity of demand, $e_d = \dfrac{p}{x}.\dfrac{dx}{dp} = \dfrac{(4-5x)^2}{x} \times \dfrac{-1}{10(4-5x)} = -\dfrac{4-5x}{10x}$.

Now, we are given that $|e_d| = 1$

$$\therefore \frac{4-5x}{10x} = 1 \qquad \text{or, } 10x = 4 - 5x \qquad \therefore 15x = 4 \qquad \therefore x = \frac{4}{15}$$ **(Ans.)**

Example 2.15 : Demand function : $q = \dfrac{6M^2}{P} + M$. Show that $1 < e_M < 2$. Also consider the range of e_p.

Solution : We have, $q = \frac{6M^2}{P} \quad M$

$$\therefore \frac{\partial q}{\partial M} = \frac{12M}{P} + 1 = \frac{12M + P}{P}$$

Now, income elasticity of demand, $e_M = \frac{M}{q} . \frac{\partial q}{\partial M} = \frac{M}{q}\left(\frac{12M + P}{P}\right)$

$$= \frac{12M^2 + MP}{P} \times \frac{P}{6M^2 + MP}$$

(as $q = \frac{6M^2}{P} + M = \frac{6M^2 + MP}{P}$)

$$= \frac{12M^2 + MP}{6M^2 + MP} = \frac{M(12M + P)}{M(6M + P)} = \frac{12M + P}{6M + P} = \frac{6M + P}{6M + P} + \frac{6M}{6M + P} = 1 + \frac{6M}{6M + P}$$

As $0 < \frac{6M}{6M + P} < 1$,

$\therefore$ e_M will be greater than 1 but less than 2, i.e., $1 < e_M < 2$ (**proved**).

Similarly, $|e_p| = \frac{P}{q} . \frac{q}{p}$

Now, $\frac{q}{P} \quad \frac{6M^2}{P^2} \quad \therefore |e_p| = \frac{P}{q} \quad \frac{6M^2}{P^2}$

$$= \frac{6M^2}{Pq} \quad \frac{6M^2}{p \; \frac{6M^2 \quad MP}{P}} = \frac{6M^2}{6M^2 + MP} = \frac{6M}{6M + P}$$

As $0 \quad \frac{6M}{6M \quad P} \quad 1$, $0 < |e_p| < 1$.

2.5.2 To Determine Marginal Values

We have mentioned how the concept of marginal is very much important in various economic decision making. Now, this marginal concept can be known just from the concept of derivative. If $y = f(x)$, then $\frac{dy}{dx}$ or $f(x)$ is the change in y due to a very small change in x. Now, if x changes by one unit, then $\frac{dy}{dx}$ or $f'(x)$ gives us the change in y due to one unit change in x. Then $\frac{dy}{dx}$ gives us the marginal value of x.

Consider some examples from Economics. We may assume that total revenue (R) of a seller depends on the volume of sales (q) i.e., R = f(q). Then $\frac{dR}{dq}$ or $f'(q)$ is the marginal revenue of selling one additional unit of output. For example, let the inverse demand function be : p = a – bq. This is actually the AR (average revenue) curve. This is because, we know, total revenue, $R \equiv p \times q$. So, averge revenue, $AR \equiv \frac{TR}{q} \equiv \frac{pq}{q} \equiv p$. Thus, average revenue is identical with price. Thus, $p \equiv AR = a - bq$ is the AR curve. Then total revenue, $R \equiv p \times q \equiv AR \times q = (a - bq)q = aq - bq^2$. Then $MR = \frac{dR}{dq} = a - 2bq$. Thus, if AR curve is linear, MR curve will also be linear. Further, if q = 0, then AR = a and also MR = a. Thus, both AR and MR curves will have same vertical intercept (= a). So they will start from the same point on the vertical axis. Further, slope of AR = $\frac{dAR}{dq} = -b$ while slope of MR curve = $\frac{dMR}{dq} = -2b = 2(-b)$. Thus, slope of MR curve will be twice of that of AR curve. Further, MR = a – 2bq = (a – bq) – bq = AR – bq.

So, $MR - AR = -bq < 0 \quad \therefore MR < AR$.

Thus, if AR is falling i.e., if AR curve is downward sloping, then MR curve will lie below the AR curve.

Similarly, from the derivative of total cost function, we can get marginal cost (MC). We may assume that total cost (C) of a firm depends on the size of output(q) i.e., C = f(q). Then marginal cost or MC = $\frac{dC}{dq}$ or $f'(q)$. For example, let the total cost function be given by : $C = a_0 + a_1q + a_2q^2 + a_3q^3$. If q = 0, $C = a_0$. So, a_0 represents total fixed cost and $TVC = a_1q + a_2q^2 + a_3q^3$.

Now, $MC = \frac{dC}{dq} = \frac{dTVC}{dq} = a_1 + 2a_2q + 3a_3q^2$.

In general, we write, C = TFC + TVC

Now, $MC = \frac{dC}{dq} = 0 + \frac{dTVC}{dq}$ (as TFC is constant and derivative of a constant is zero).

Thus, MC = $\frac{dC}{dq} = \frac{dTVC}{dq}$ i.e., MC is the change in total cost or change in total variable cost due to one unit change in output (as TFC component of total cost is fixed). Similarly, by taking derivative of total utility function, we get marginal utility (MU) of a commodity. If the total utility function is : U = f(q), then marginal utility is given by $\frac{dU}{dq}$ or f′(q). Similarly, by differentiating total product function with respect to a particular input, we get marginal product of that input. For example, let the total product (q) function be : q = f(L). Then marginal product of labour is given by $\frac{dq}{dL}$ or f′(L). If we take a total product function of more general form : q = f(L, K), then its partial derivatives $\frac{\delta q}{\delta L}$ (or f_L) and $\frac{\partial q}{\partial K}$ (or f_K) will give us marginal products of labour (L) and capital (K) respectively.

2.5.3 To Determine Profit-Maximising and Cost-Minimising Output

The concept of derivative is also useful to determine profit maximising output and cost minimising output. Consider first the case of profit. We know that total profit (π) is the difference between total revenue(R) and total cost(C). Again, both total revenue and total cost may be assumed to be functions of the level of output. Thus, total profit, π = R – C = R(q) – C(q). So, π = f(q) i.e., total profit is a function of output. Now, to determine the level of output at which profit is maximum, we have to fulfil two conditions : (i) $\frac{d\pi}{dq}$ or f′(q) = 0 and $\frac{d}{dq}\left(\frac{d\pi}{dq}\right)$ or $\frac{d^2\pi}{dq^2}$ or f″(q) < 0. Thus, to determine profit-maximising output, we have to consider the first derivative of the profit function and then the derivative of the first derivative i.e., the second derivative of the profit function.

Consider now the case of cost. Let the total cost function be : C = f(q) where q is the level of output. Then average cost, $AC = \frac{C}{q} = \frac{f(q)}{q}$. Thus, AC is also a function of output (q) i.e., AC = h(q).

Now, to minimise AC, two conditions are to be fulfilled : (i) $\frac{dAC}{dq}$ or h′(q) = 0 and

(ii) $\frac{d}{dq}\left(\frac{dAC}{dq}\right)$ or $\frac{d^2AC}{dq^2}$ or, $h''(q) > 0$. Thus, to determine AC-minimising output, we have to apply the concept of derivative.

The issues of maximisation and minimisation have been considered in details in the next unit.

2.5.4 To Determine Slope and Curvature of Indifference Curve, Isoquant, etc.

To determine the slope and curvature of indifference curve, isoquant, etc. we need the help of derivation. First we consider the case of an indifference curve. Let the utility function be : $U = f(q_1, q_2)$ where q_1 and q_2 are the quantities of two goods, Q_1 and Q_2, respectively. Now, taking total derivative of the utility function,

we get, $dU = \frac{\partial f}{\partial q_1}.dq_1 + \frac{\partial f}{\partial q_2}.dq_2$

Using alternative notation, $\frac{\partial f}{\partial q_1} = f_1(=MU_1)$ and $\frac{\partial f}{\partial q_2} = f_2(= MU_2)$, we get, $dU = f_1dq_1 + f_2dq_2$.

Now, along a given indifference curve, utility level is constant, say, U_0. So, the equation of a particular indifference curve is : $U_0 = f(q_1, q_2)$. As U is fixed at U_0 along an indifference curve, $dU = 0$, So, we have,

$f_1dq_1 + f_2dq_2 = 0$ or, $f_2dq_2 = - f_1dq_1$

$$\therefore \frac{dq_2}{dq_1} = -\frac{f_1}{f_2} = -\frac{MU_1}{MU_2}.$$

Under the assumption that $MU_1(= f_1) > 0$ and $MU_2(= f_2) > 0$, $\frac{dq_2}{dq_1} = -\frac{f_1}{f_2} < 0$.

Now, $\frac{dq_2}{dq_1}$ is the slope of an indifference curve. Hence, an indifference curve will be negatively sloped. The expression, $-\frac{dq_2}{dq_1}$ is called the marginal rate of substitution (MRS). Thus, MRS = $-\frac{dq_2}{dq_1} = \frac{f_1}{f_2} = \frac{MU_1}{MU_2}$.

To know the curvature of the indifference curve, we have to take further derivative of $\frac{dq_2}{dq_1}$ i.e., $\frac{d}{dq_1}\left(\frac{dq_2}{dq_1}\right)$ or $\frac{d^2q_2}{dq_1^2}$ and to consider its sign.

We have, $\frac{dq_2}{dq_1} = -\frac{f_1}{f_2}$. We shall keep in mind that in indifference curve analysis, utility functions are interdependent. So, MU_1 (or f_1) and MU_2(or f_2) both will depend on q_1 and q_2 i.e., $f_1 = f_1(q_1, q_2)$ and $f_2 = f_2(q_1, q_2)$.

Thus, $\frac{dq_2}{dq_1} = -\frac{f_1}{f_2} = -\frac{f_1(q_1, q_2)}{f_2(q_1, q_2)}$

Now we differentiate $\frac{dq_2}{dq_1}$ with respect to q_1.

Then, $\frac{d}{dq_1}\left(\frac{dq_2}{dq_1}\right) = \frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^2}\left[f_{11}f_2 + f_{12}\frac{dq_2}{dq_1}.f_2 - f_{21}f_1 - f_{22}\frac{dq_2}{dq_1}.f_1\right]$

$$= -\frac{1}{f_2^2}\left[f_{11}f_2 + f_{12}\left(-\frac{f_1}{f_2}\right).f_2 - f_{21}f_1 - f_{22}\left(-\frac{f_1}{f_2}\right).f_1\right]$$

$$= -\frac{1}{f_2^2}\left[f_{11}f_2 - f_{12}f_1 - f_{21}f_1 + f_{22}.\frac{f_1^2}{f_2}\right]$$

$$= -\frac{1}{f_2^3}\left[f_{11}f_2^2 - f_{12}f_1f_2 - f_{21}f_1f_2 + f_{22}f_1^2\right]$$

Now, from Young's theorem we know that cross partial derivatives are equal i.e., $f_{12} = f_{21}$. So, we get,

$$\frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^3}\left[f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2\right]$$

Now, if we assume that MRS is diminishing, then it implies that $\frac{d^2q_2}{dq_1^2} > 0$. Then the braketed portion of the RHS is negative. In this case, the indifference curve will be strictly convex to the origin. If $\frac{d^2q_2}{dq_1^2} < 0$, then MRS is increasing and the IC will be strictly concave to the origin. Again, if $\frac{d^2q_2}{dq_1^2}$ = 0, then the indifference curve will be linear.

In the same manner we may consider the slope and curvature of an iso-quant. Let the production function be = $Y = f(x_1, x_2)$ whose x_1 and x_2 are the quantities of two inputs, X_1 and X_2 respectively. In that case, if Y is fixed at Y_0 the equation of a given isoquant is : $Y_0 = f(x_1, x_2)$. Now proceeding in the earlier manner, we can deduce that $\frac{dx_2}{dx_1} = -\frac{f_1}{f_2}$ where f_1 and f_2 are the marginal productivities of two inputs, X_1 and X_2 respectively, i.e., $f_1 = \frac{\partial f}{\partial x_1} = MP_1$ and $f_2 = \frac{\partial f}{\partial x_2} = MP_2$. The expression $-\frac{dx_2}{dx_1}$ is called the marginal rate of technical substitution (MRTS) between the two inputs. Thus, MRTS = $-\frac{dx_2}{dx_1} = \frac{f_1}{f_2} = \frac{MP_1}{MP_2}$. We see that $\frac{dx_2}{dx_1} = -\frac{f_1}{f_2} = -\frac{MP_1}{MP_2} < 0$ as MP_1(or f_1) and MP_2(or f_2) are positive. Thus, the slope of the isoquant is negative or the iso-quant is negatively sloped.

To know the curvature of the isoquant, we have to differentiate $\frac{dx_2}{dx_1}$ further with respect to x_1 i.e., we have to know the value of $\frac{d}{dx_1}\left(\frac{dx_2}{dx_1}\right)$ or $\frac{d^2x_2}{dx_1^2}$. Proceeding in the same manner as in the case of indifference curve, we get, $\frac{d^2x_2}{dx_1^2} = -\frac{1}{f_2^3}(f_{11}f_2^2 - 2f_{12} + f_{22}f_1^2)$ assuming $f_1 = f_1(x_1, x_2)$ and $f_2 = f_2(x_1, x_2)$ and putting $f_{12} = f_{21}$.

Now, if we assume that MRTS is diminishing, then $\frac{d^2x_2}{dx_1^2} > 0$. In that case, the iso-quant will be strictly convex. If $\frac{d^2x_2}{dx_1^2} = 0$, the isoquant will be linear. If $\frac{d^2x_2}{dx_1^2} < 0$, the isoquant will be concave to the origin. Thus, to know the slope and curvature of an indifference curve or of an isoquant or of any curve, we have to use the concept of derivative.

2.6 Relation between Price Elasticity of Demand and Total Expenditure or Total Revenue

What is total expenditure (TE) to the buyer is total revenue (TR) to the seller. They are, by difinition, equal to each other. TE or TR = p × q where p = price per unit of a commodity and q is the amount of the commodity bought or sold. We know that quantity demanded (q) is a function of price (p). So, total expenditure, TE = p.f(p) = E(p). Thus, if p changes, TE may change. If p falls, then from the law of demand we know that q will rise $\left(\frac{dq}{dp} < 0\right)$. So, TE (= p × q) may rise, remain constant or fall. That depends on the relative rates of change in demand and change in price. In other words, whether TE will rise or not due to change in p, depends on the value of price elasticity of demand.

We know that $e_d = \frac{p}{q}.\frac{dq}{dp}$. As $\frac{dq}{dp} < 0$, $e_d = \frac{p}{q}.\frac{dq}{do} < 0$. So, the absolute value of e_d, $|e_d| = -\frac{p}{q}.\frac{dq}{dp}$.

Let us see what happens to TE or simply, E due to change in p. That can be known by differentiating E(= p × q) with respect to p. So, we get, $\frac{dE}{dp} = q + p.\frac{dq}{dp}$.

Now if TE remains the same due to change in p, then $\frac{dE}{dp} = 0$

So, $q + p.\frac{dq}{dp} = 0$, or $p.\frac{dq}{dp} = -q$ or, $-\frac{p}{q}.\frac{dq}{dp} = 1$ or, $|e_d| = 1$

Its converse is also true, i.e., if $|e_d| = 1$, then total expenditure of the buyer will remain the same due to change in price.

In the same manner, it can be proved that if total expenditure rises as price falls, or if total expenditure falls as price rises, then $|e_d| > 1$ i.e., demand is elastic. Its converse is also true i.e., if demand is elastic, then total expenditure will rise with fall in price and will fall with the rise in price.

Similary, consider the opposite case. If total expenditure falls with the fall in price or rises with the rise in price, then $|e_d| < 1$ i.e., demand is inelastic. Its converse is also true i.e., if demand is inelastic, then total expenditure will fall with the fall in price and will rise with the rise in price.

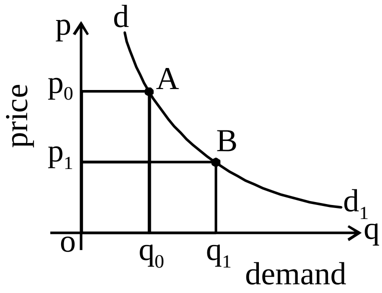
2.6.1 Case of Constant Expenditure or Outlay Curve

We know that total expenditure, E = p × q where q = f(p). If expenditure remains the same due to change in p, then $\frac{dE}{dp} = 0$

So, $q + p.\frac{dq}{dp} = 0$ or, $p.\frac{dq}{dp} = -q$ or, $|e_d| = 1$

Thus, if expenditure of the buyer remains the same, then $|e_d| = 1$ i.e., demand is unit elastic. We like to know the shape of this unit elastic demand curve or constant expenditure (or outlay) curve. Consider the diagram 2.1 in which we have drawn a demand curve dd_1. We take any point A on this demand curve. At this point, $p = op_0$ and $q = oq_0$. So, TE of the consumer = pq = $op_0 \times oq_0 = \square\ op_0Aq_0$.

Thus, total expenditure at any point on the demand curve is given by the area of the rectangle obtained by drawing two perpendiculars on the two axes. Similarly, total expenditure of the buyer at B = p × q = $op_1 \times oq_1 = \square\ op_1Bq_1$. Now, we know that if $|e_d| = 1$, then TE remains the same i.e., TE is constant. That is, in our figure, Area of $\square\ op_0Aq_0$ = Area of $\square\ op_1Bq_1$. And this will hold for any point on the demand curve, dd_1. Thus, our demand curve in this case will be such that the area of all the rectangles under this curve is the same or constant. Such a curve is called a rectangular hyperbola. Thus, if $|e_d| = 1$, the demand curve will be a rectangular hyperbola. In our figure, dd_1 is a rectangular hyperbola. On this curve, $|e_d| = 1$ and expenditure is constant. Hence it is called unit elastic demand curve or constant expenditure (outlay) curve. This curve will asymptote to the axes but will never meet the axes. Such a curve is also called an asymptotic curve. As expenditure on this curve is constant, its equation will be : pq = k or, $q = \frac{k}{p}$ or, $q = kp^{-1}$ where k is a constant.



(Fig. 2.1)

2.7 Relation among AR, MR and Price Elasticity of Demand

There is a standard relation among AR, MR and price elasticity of demand. To deduce that relation, we have to use the concept of derivative. Let us consider it.

We know that total revenue, TR = p × q where p = price and q = quantity sold. We know from inverse demand function, p = f(q). So, R = p × q = f(q) × q. So, total revenue R is function of q. Now, in order to deduce the relation among AR, MR and price elasticity of demand (e_d), let us define them first. AR or average revenue is the revenue per unit of output sold. If R is the total revenue from sales of q unit of output, then average revenue, $AR \equiv \frac{\text{total revenue}}{\text{units of output sold}} \equiv \frac{R}{q} \equiv \frac{p \times q}{q} \equiv p$. Average revenue is thus identical with price ($AR \equiv p$). Marginal revenue (MR) is the change in total revenue if sales change by one unit. In terms of calculus, $MR = \frac{dR}{dq}$. Thus, MR can be obtained from first derivative of the total revenue function with respect to q. Price elasticity of demand (e_d) is the percentage change in quantity demanded due to one percent change in price, *ceteris paribus*. Thus,

$$e_d = \frac{\text{percentage change in quantity demanded}}{\text{percentage change in price}} = \frac{\frac{dq}{q} \times 100}{\frac{dp}{p} \times 100} = \frac{dq}{q} \times \frac{p}{dp} = \frac{p}{q} . \frac{dq}{dp}.$$

Now, the law of demand states an inverse relation between price and quantity demanded. So, $\frac{dq}{dp} < 0$.

Hence, $e_d = \frac{p}{q} . \frac{dq}{dp} < 0$. So, the absolute value of e_d is given by, $|e_d| = -\frac{p}{q} . \frac{dq}{dp}$.

Let us deduce the standard relation among AR, MR and e_d. We have, total revenue, $R = p \times q$ where $p = f(q)$. Now differentiating both sides with respect to q, we get,

$$\frac{dR}{dq} = p + q.\frac{dp}{dq} = p\left(1 + \frac{q}{p} . \frac{dp}{dq}\right) \qquad \text{or, } MR = p\left(1 - \frac{1}{-\frac{p}{q} . \frac{dq}{dp}}\right) = p\left(1 - \frac{1}{|e_d|}\right)$$

$$\therefore MR = p\left(1 - \frac{1}{|e_d|}\right) = AR\left(1 - \frac{1}{|e_d|}\right) \text{ as } AR \equiv p.$$

This is our standard relation among AR($\equiv$ p), MR and price elasticity of demand (e_d). We have obtained this relation by using derivative of R with respect to q.

From this relation we can determine the value of one variable if the values of other two are given. From the relation, we can determine MR if AR(or p) and $|e_d|$ are given. Again, we can write, AR or $p = \frac{|e_d|}{|e_d| - 1}$. MR. Again, $|e_d| = \frac{AR - MR}{AR}$.

2.7.1 Relation among TR, MR and Price Elasticity of Demand

There is a standard relation among AR, MR and e_d or price elasticity of demand. The relation is : $MR = AR\left(1-\frac{1}{|e_d|}\right) = p\left(1-\frac{1}{|e_d|}\right)$. From this relation, we can easily mention the relation among TR, MR and price elasticity of demand. We know that MR is the addition to total revenue. So, when MR > 0, TR will rise. Similarly, so long MR < 0, TR will fall. If MR = 0, TR will remain constant. Now, from the above relation among AR, MR and $|e_d|$, we see that above relation among AR, MR and $|e_d|$, we see that $MR \gtreqless 0$ according as $|e_d| \gtreqless 1$. From this we can say the following : When $|e_d| > 1$, MR > 0 and TR will rise with the rise in q or fall in p.

If $|e_d| < 1$, MR < 0 and TR will fall with the rise in q or fall in p.

If $|e_d| = 1$, MR = 0, and so TR will remain the same due to rise or fall in p or q.

Thus, we can make the following statements :

1. If demand is elastic ($|e_d| > 1$), a fall in price or a rise in q will lead to a rise in total revenue, while a rise in price or fall in q will lead to fall in TR.

2. If demand is inelastic ($|e_d| < 1$), a fall in price or a rise in demand leads to a fall in TR, while a rise in price or a fall in q will lead to a rise in TR.

3. If demand is unitary elastic ($|e_d| = 1$), TR will remain unchanged for a change in price or quantity.

2.8 Elasticity of Factor Substitution and Shape of Isoquant

We know that elasticity is a measure of the percentage change in one variable in respect of a percentage change in another variable. If y = f(x), then elasticity of y with respect to x is given by,

$$e_x = \frac{\text{percentage change in y}}{\text{percentage change in x}} = \frac{\frac{dy}{y}\times 100}{\frac{dx}{x}\times 100} = \frac{x}{y}.\frac{dy}{dx}.$$

We see that to measure elasticity, we have to know $\frac{dy}{dx}$ i.e., the derivative of y with respect to x. Hence to measure elasticity of factor substitution also, we have to apply the concept of derivative.

Let us try to clarify first the concept of elasticity of factor substitution. Let the production function be : q = F(K.L) where q = quantity of output and K and L are the amounts of capital and labour, respectively. Along a particular iso-quant, output(q) is fixed, say, at q_0. So, the equation of a particular isoquant is $q_0 = F(K, L)$. On this isoquant, the firm will be in equilibrium where its cost to produce that output is minimum. This is attained at the point where $\frac{MP_L}{MP_K} = \frac{p_L}{p_K}$ i.e., MRTS = $\frac{p_L}{p_K}$.

Now, this factor combination will change if the relative factor price of the inputs changes. The elasticity of factor substitution measures the responsiveness of the optimal factor-combination to a change in the relative prices of the two inputs. In other words, we may say that $K/L = f(p_L/p_k)$. The input ratio (K/L) will change if the relative factor price of the two inputs (p_L/p_k) changes. Hence, the elasticity of factor substitution may be expressed as,

$$\sigma = \frac{\text{percentage change in K/L}}{\text{percentage change in } p_L / p_K}$$

where K/L is the optimal capital-labour ratio and p_L and p_K are the prices of labour and capital, respectively.

Now, in equilibrium factor combination, $\frac{MP_L}{MP_K} = \frac{p_L}{p_K}$ = MRTS.

So, the elasticity of substitution can be expressed as,

$$\sigma = \frac{\text{percentage change in K/L}}{\text{percentage change in MRTS}}$$

Thus, elasticity of factor substitution measures the percentage change in factor proportion due to one percent change in the marginal rate of technical substitution (MRTS). Now, putting MRTS = MP_L/MP_K, we have,

$$\sigma = \frac{\frac{d(K/L)}{K/L}}{\frac{d(MP_L/MP_K)}{MP_L/MP_K}} = \frac{d\log(K/L)}{d\log(MP_L/MP_K)} = \frac{(MP_L/MP_K)}{K/L}.\frac{d(K/L)}{d(MP_L/MP_K)}$$

Thus, to know elasticity of factor substitution, we have to apply the concept of derivative.

Alternatively, $\sigma = \frac{d\log(K/L)}{d\log(MRTS)}$

In general, σ is finite implying convexity of isoquants. Higher value of σ implies higher degree of substitution between the two inputs i.e., less will be the convexity of isoquants, and *vice versa.*

In one extreme, if $\sigma = \propto$, there is infine possibility of substitution and isoquants will be linear. For a linear isoquant, its slope is constant and hence (MP_L/MP_K) is constant i.e., $d(MP_L/MP_K) = 0$ and hence (MP_L/MP_K) is constant i.e., $d(MP_L/MP_K) = 0$ and so $\sigma = \propto$. In another extreme, $\sigma = 0$ i.e., there is no possibility of factor substitution. In this case, iso-quants are L-shaped or right angled. The firm will employ two inputs in a given ratio. Then (K/L) is constant i.e., $d(K/L) = 0$ and hence $\sigma = 0$. The same discussion is applicable to the case of indifference curve also.

2.9 Homogeneous Production Function

Let our production function be $q = F(K, L)$ where q = quantity of output and K and L are amounts of capital and labour, respectively. Now, we know that a function is said to be homogeneous of degree n if multiplication of each independent variable by a constant λ will change the value of the dependent variable by the proportion λ^n. The value of n is called the degree of homogeneity.

So, in our context, the production function $q = F(K, L)$ is said to be homogeneous of degree n if $F(\lambda K, \lambda L) = \lambda^n.q$. A homogeneous production function possesses some important properties. We consider some of those properties below.

2.9.1 Homogeneous Production Function and its Properties

Property 1 : If the production function is homogeneous of degree n, then the marginal productivities of the inputs will be homogeneous of degree (n – 1).

Proof : Let our production function be : $q = F(K, L)$. We assume that this function is homogeneous of degree n. So, by definition, $\lambda^n.q = F(\lambda K, \lambda L)$

Putting $\lambda = \frac{1}{L}$, we get, $\left(\frac{1}{L}\right)^n.q = F\left(\frac{K}{L}, 1\right)$

$\therefore$ $q = L^n.f(K/L)$ ... (1) where $f(K/L) = F(K/L, 1)$.

Now, marginal productivity of capital and labour are given by $\frac{\partial q}{\partial k}$ (= f_k) and $\frac{\partial q}{\partial L}$ (= f_L), respectively. We first calculate $\frac{\partial q}{\partial k}$ or MP_k from (1)

$$MP_k = \frac{\partial q}{\partial K} = L^n.f'\left(\frac{K}{L}\right).\frac{d}{dk}\left(\frac{K}{L}\right)$$

$$\text{or, } MP_k = L^n.\ f'\left(\frac{k}{L}\right).\ \frac{1}{L} = L^{n-1}.f'\left(\frac{K}{L}\right) \qquad \text{... (2)}$$

Similarly we can calculate MP_L or $\frac{\delta q}{\delta L}$.

$$MP_L = \frac{\delta q}{\delta L} = n.L^{n-1}.f\left(\frac{K}{L}\right) + L^n.f'\left(\frac{K}{L}\right).\frac{d}{dL}\left(\frac{K}{L}\right)$$

$$= nL^{n-1}.f\left(\frac{K}{L}\right) + L^n.f'\left(\frac{K}{L}\right)\left(-\frac{K}{L^2}\right)$$

$$\therefore MP_L = nL^{n-1}.f\left(\frac{K}{L}\right) - L^{n-1}.f'\left(\frac{K}{L}\right)\left(\frac{K}{L}\right)$$

$$\text{or, } MP_L = L^{n-1}\left[n.f\left(\frac{K}{L}\right) - \frac{K}{L}.f'\left(\frac{K}{L}\right)\right] \qquad \text{... (3)}$$

Let us consider the degree of homogeneity of MP_k and MP_L. To do this, we increase both K and L and by λ times and see how the values of MP_K and MP_L change. We have $MP_K = L^{n-1}.f'\left(\frac{K}{L}\right)$. When K and L both are increased by λ times, the new value of MP_K, say, $MP_K^* = (\lambda L)^{n-1}.\ f'\left(\frac{\lambda K}{\lambda L}\right)$

$$\text{or, } MP_K^* = \lambda^{n-1}.\left[L^{n-1}.f'\left(\frac{K}{L}\right)\right]$$

or, $MP_K^* = \lambda^{n-1}.MP_K$.

Similarly, we have, $MP_L = L^{n-1}\left[nf\left(\frac{K}{L}\right) - \frac{K}{L}.f'\left(\frac{K}{L}\right)\right]$

When both K and L are increased by λ times, the new value of MP_L, say, MP_L^* becomes,

$$MP_L^* = (\lambda L)^{n-1}\left[nf\left(\frac{\lambda K}{\lambda L}\right) - \frac{\lambda K}{\lambda L}f'\left(\frac{\lambda K}{\lambda L}\right)\right] = \lambda^{n-1}.L^{n-1}\left[nf\left(\frac{K}{L}\right) - \frac{K}{L}f'\left(\frac{K}{L}\right)\right]$$

or, $MP_L^* = \lambda^{n-1}.\ MP_L$

Thus we see that if we change both K and L by λ times, MP_K and MP_L will change by λ^{n-1} times. So MP_K and MP_L are homogeneous of degree (n – 1) if the original production function q = F(K, L) is homogeneous of degree n.

We may get an important corollary of this property. If the degree of homogeneity of the production function is one (n = 1), then the marginal productivity of its inputs will be homogeneous of degree zero (n – 1 = 1 – 1 = 0). In other words, if the production function is homogeneous of degree 1 and if the inputs are increased or decreased by a certain rate, their marginal productivities will remain unchanged. In fact, in this case, marginal productivities of the inputs will depend on the input ratio. When both the inputs are changed by λ times, the value of input ratio remains unchanged and hence marginal productivities remain unchanged. We have considered this in the next property.

Property 2 : If the production function is homogeneous of degree 1, then its marginal productivities will be homogeneous of degree zero, or the marginal productivities will depend only on input ratio.

Proof : Let our production function q = F(K, L) be homogeneous of degree 1. So by definition of homogeneous function, we can write,

$\lambda^1 q = F(\lambda K, \lambda L)$. Thus, $\lambda q = F(\lambda K, \lambda L)$.

When n = 1, the production function is also called linearly homogeneous.

We now put $\lambda = \frac{1}{L}$.

So, $\frac{q}{L} = F\left(\frac{K}{L}, 1\right)$ or, $q = L.f\left(\frac{K}{L}\right)$ where $F\left(\frac{K}{L}, 1\right) = f\left(\frac{K}{L}\right)$

Now, $MP_K = \frac{\delta q}{\delta K} = L.f'\left(\frac{K}{L}\right).\frac{d}{dK}\left(\frac{K}{L}\right) = L.\, f'\left(\frac{K}{L}\right).\frac{1}{L} = f'\left(\frac{K}{L}\right)$

$$MP_L = \frac{\delta q}{\delta L} = f\left(\frac{K}{L}\right) + L.f'\left(\frac{K}{L}\right).\frac{d}{dL}\left(\frac{K}{L}\right) = f\left(\frac{K}{L}\right) + L.f'\left(\frac{K}{L}\right)\left(-\frac{K}{L^2}\right)$$

or, $MP_L = f\left(\frac{K}{L}\right) - \frac{K}{L}.f'\left(\frac{K}{L}\right)$

We see that both MP_K and MP_L are functions of or depend on $\frac{K}{L}$ i.e., on input ratio.

Now, if both K and L are changed by λ times, the value of input ratio i.e., value of $\frac{K}{L}$ will remain unchanged. Hence the values of MP_K and MP_L will remain unchanged or mathematically, they will change by λ^0 times. In other words, marginal productivities will be homogeneous of degree zero in this case. Let us formally show it.

Our $MP_K = f'\left(\frac{K}{L}\right)$ and $MP_L = f\left(\frac{K}{L}\right) - \frac{K}{L}.f'\left(\frac{K}{L}\right)$. When K and L are changed by λ

times, the new value of MP_K, say, $MP_K^* = f'\left(\frac{\lambda K}{\lambda L}\right) = f'\left(\frac{K}{L}\right) = MP_K = \lambda^0.MP_K$.

Similarly, the new value of MP_L, say MP_L^* is given by :

$$MP_L^* = f\left(\frac{\lambda K}{\lambda L}\right) - \frac{\lambda K}{\lambda L}.f'\left(\frac{\lambda K}{\lambda L}\right) = f\left(\frac{K}{L}\right) - \frac{K}{L}.f'\left(\frac{K}{L}\right) = MP_L = \lambda^0.MP_L.$$

Thus, if the production function is homogeneous of degree 1 or linearly homogeneous, then its marginal productivities will be homogeneous of degree zero or its marginal productivities will then depend only on the input ratio.

Property 3 : If the production function q = F(K, L) is homogeneous of degree 1, then $F_{LL}.L + F_{LK}.K = 0$, and $F_{KL}.L + F_{KK}K = 0$

Proof : Our production function q = F(K, L) is homogeneous of degree 1 (or linearly homogeneous). Now, from Euler's theorem we know that if a function Z = f(x, y), then

$x.\frac{\delta f}{\partial x} + y.\frac{\partial f}{\partial y} = 1.z = z$. i.e., $x.f_x + yf_y = z$. (see section 1.11 of unit 1)

So, applying this Euler's theorem we can write, $K.F_K + L.F_L = q$ where $F_K = \frac{\delta q}{\partial K}$ or $\frac{\partial F}{\partial K}$ and $F_L = \frac{\delta q}{\partial K}$ or $\frac{\partial F}{\partial L}$ i.e., F_K and F_L are marginal productivities of K and L, respectively. Now, differentiating this function partially with respect to K, we get,

$$F_K.1 + K.F_{KK} + L.F_{LK} = \frac{\delta q}{\delta K} = F_K$$

$\therefore$ K. $F_{KK} + L.F_{LK} = 0$ (proved)

Similarly, differentiating with respect to L,

$K.F_{KL} + F_L.1 + L.F_{LL} = \frac{\delta q}{\partial L} = F_L$ or, $K.F_{KL} + L.F_{LL} = 0$ (proved)

2.9.2 Homogeneous Production Function and Returns to Scale

The concept of homogeneous production function may be used to show different concepts of returns to scale. In order to show this, we first explain the concept of returns to scale. A changed in the scale of production means that the amounts of all inputs or factors are changed in the same proportion. Returns to scale refers to changes in output level as a result of changes in scale. Now the law of returns to scale may be of three types : (i) constant returns to scale, (ii) increasing returns to scale and (iii) decreasing returns to scale.

If the level of output rises in the same proportion in which inputs are increased, there will be constant returns to scale (CRS). If output level rises at a greater rate than inputs or at a greater rate than the change in scale, there will be increasing returns to scale (IRS). Again, there will be decreasing returns to scale (DRS) if output level rises at a lower rate than inputs or than the change in scale.

The concepts of three types of returns to scale can be explained with the help of homogeneous production function. The production function q = F(K, L) is said to be homogeneous of degree n if $F(\lambda K, \lambda L) = \lambda^n.Y$. This means that when K and L are increased by λ times, total output will increase by λ^n times. The constant n is called the degree of homogeneity. Now, if n = 1, then $\lambda Y = F(\lambda K, \lambda L)$. This means that if both the inputs K and L are increased by λ times, output level will increase by λ times. This implies that there are constant returns to scale (CRS). Thus, if the degree of homogeneity of a production function is unity, the function will exhibit constant returns to scale (CRS). This is also known as homogeneous production function of degree 1 or linearly homogeneous production function. Thus, if the production function is homogeneous of degree one (n = 1) or linearly homogeneous, there will be CRS. If n > 1, i.e., the degree of homogeneity is greater than one, there will be IRS. Similarly, if n < 1, there will be DRS. Thus, all three types of returns to scale can be expressed by the degree of homogeneity of a homogeneous production function.

Let us consider some examples.

Example 2.16 : Let the production function be : q = 3K + 4L. Determine the type of returns to scale of this production function,

Solution : Here we have, q = 3K + 4L.

Now we increase both K and L by λ times. The new level of output = q*(say) = $3(\lambda K) + 4(\lambda L) = \lambda(3K + 4L) = \lambda.q = \lambda^1.q$.

Thus the given production function is homogeneous of degree 1 (linearly homogeneous). As the degree of homogeneity is equal to one, the given function displays constant returns to scale (CRS).

Example 2.17 : Determine the type of returns to scale of the production function, $Y = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$.

Solution : We have, $Y = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$.

Now, if we increase x_1 and x_2 by λ times. The new level of output, say, Y* = $(\lambda x_1)^{\frac{1}{4}}(\lambda x_2)^{\frac{1}{4}} = \lambda^{\frac{1}{4}+\frac{1}{4}}.x_1^{\frac{1}{4}}x_2^{\frac{1}{4}} = \lambda^{\frac{1}{2}}.Y$ Thus, the given production function is homogeneous of degree $\frac{1}{2}$. Here, as the degree of homogeneity is less than 1, the given function displays decreasing returns to scale.

Example 2.18. Determine the degree of homogeneity of the production function, $z = x^2 + y^2$ and interpret your result.

Solution : We have, $z = x^2 + y^2$

Now, we increase both x and y by λ times. The new output level, say, $z^* = (\lambda x)^2 + (\lambda y)^2 = \lambda^2(x^2 + y^2) = \lambda^2.z$

So, the degree of homogeneity = 2. As the degree of homogeneity is greater than one, the given production function exhibits increasing returns to scale (IRS).

Example 2.19 : Examine the type of returns to scale if the production function is : $q = 30K^{\alpha}L^{1-\alpha}(0 < \alpha < 1)$.

Solution : The given production function is : $q = 30K^{\alpha} L^{1-\alpha}$. Now we increase both K and L by λ times. The new output level, say, $q^* = 30(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = \lambda^{\alpha+1-\alpha} 30K^{\alpha}L^{1-\alpha} = \lambda^1.q$.

Thus, the given production function is homogeneous of degree 1 or linearly homogeneous. So the given production function displays constant returns to scale (CRS).

Example 2.20 : What type of returns to scale will operate if the production function is : $q = 10\sqrt{K} + 20\sqrt{L}$?

Solution : The given production function is : $q = 10\sqrt{K} + 20\sqrt{L}$. We now increase both K and L by λ times. The new output level,

$$q^* \text{ (say)} = 10\sqrt{\lambda K} + 20\sqrt{\lambda L} = \sqrt{\lambda}(20\sqrt{K} + 20\sqrt{L}) = \lambda^{\frac{1}{2}}q.$$

Here the degree of homogeneity is $\frac{1}{2}$ which is less than one. So, the given production function is subject to decreasing returns to scale (DRS).

Example 2.21 : Determine the degree of homogeneity of the production function : $q = AK^{\alpha}L^{\beta}$ and indicate the type of returns to scale exhibited by this function.

Solution : Our production function is : $q = AK^{\alpha}L^{\beta}$.

In order to determine the degree of homogeneity of this function, we increase both K and L by λ times. The new value of q, say, $q^* = A(\lambda K)^{\alpha}(\lambda L)^{\beta} = \lambda^{\alpha+\beta}.AK^{\alpha}L^{\beta} = \lambda^{\alpha+\beta}q$.

So the given production function is homogeneous of degree $(\alpha + \beta)$.

If $(\alpha + \beta) = 1$, it will display constant returns to scale.

If $(\alpha + \beta) > 1$, it will display increasing returns to scale.

If $(\alpha + \beta) < 1$, it will display decreasing returns to scale.

2.9.3 Homogeneous Production Function and Product Exhaustion Theorem

The product exhaustion theorem states that if the production function is homogeneous of degree 1 (or subject to CRS) and if the factors are paid according to their marginal productivities, then total product will just be exhausted. There will neither be any surplus nor any deficit. This product exhaustion theorem is a corollary of Euler's theorem. This is an application of Euler's theorem in section 1.11. in Unit 1. Let us try to remember it. The Euler's theorem states that if a function z = F(x, y) is homogeneous of degree n, then $\frac{\partial f}{\partial x}.x + \frac{\partial f}{\partial y}.y = nz$

Or using simpler notation, $f_x.x + f_y y = nz$

We apply this theorem in the case of production where the production function is, q = F(K, L) where q = output, K = capital and L = Labour. If this production function is homogeneous of degree n, then by Euler's theorem, $\frac{\partial F}{\partial K}.K + \frac{\partial F}{\partial L}.L = nq$.

Let us consider what happens if n = 1 i.e., degree of homogeneity is equal to one or the production function is subject to CRS. Then we have, $\frac{\partial F}{\partial K}.K + \frac{\partial F}{\partial L}.L = q$

Now, $\frac{\partial F}{\partial K} = MP_K$ and $\frac{\partial F}{\partial L} = MP_L$

So, we can write, $MP_K.K + MP_L.L = q$

Now, if the factors of production are paid according to their marginal productivities, then $MP_K = p_K$ and $MP_L = p_L$. So, we get, $p_K.K + p_L.L = q$. Here, $p_K.K$ is the payment to capital while $p_L.L$ is the payment to labour. Thus, the LHS of the equation is total factor payment while the R.H.S. is the total output. Thus the equation implies that if the production function is homogeneous of degree 1 (or subject to CRS) and if the factors are paid according to their marginal productivities, then the total product will just be exhausted. There will neither be any surplus of total product nor there will be any deficit. This is known as product exhaustion theorem or adding up problem. This theorem actually follows from the Euler's theorem and hence it may be regarded as a corollary of the Euler's theorem.

Let us prove the product exhaustion theorem. We have the production function, q = F(K, L). It is assumed that this function is homogeneous of degree one or subject to constant returns to scale. As the production function is homogeneous of degree 1, we can write, $\lambda q = F(\lambda K, \lambda L)$

Putting $\lambda = \frac{1}{L}$, we get, $\frac{q}{L} = F\left(\frac{K}{L}, 1\right)$

or, $q = L.\ F\left(\frac{K}{L}, 1\right) = L.\, f\left(\frac{K}{L}\right)$ where $F\left(\frac{K}{L}, 1\right) = f\left(\frac{K}{L}\right)$.

Thus, we have, $q = L.\, f\left(\frac{K}{L}\right)$ ...(1)

Now, differentiating equation (1) with respect to K, we get,

$$\frac{\partial q}{\partial K} = f_K = L.f'\left(\frac{K}{L}\right).\frac{1}{L} = f'\left(\frac{K}{L}\right)$$

Multiplying both sides by K, we get, $K.\frac{\partial q}{\partial K} = K.f'\left(\frac{K}{L}\right)$ ... (2)

Again, differentiating equation (1) with respect to L,

we get, $\frac{\delta q}{\delta L} = f_L = 1.f\left(\frac{K}{L}\right) + L, f'\left(\frac{K}{L}\right)\left(-\frac{K}{L^2}\right)$

Multiplying both sides by L, we get, $L.\frac{\delta q}{\delta L} = L.f\left(\frac{K}{L}\right) - K.f'\left(\frac{K}{L}\right)$ ...(3)

Adding (2) and (3), we get, $\frac{\delta q}{\partial K}.K + \frac{\delta q}{\partial L}.L = L.f\left(\frac{K}{L}\right)$ = q [from (1)]

Or, using different notations, $f_K.K + f_L.L = q$

This is our product exhaustion theorem.

We should note one thing. Our general theorem is : $K.\frac{\delta q}{\delta K} + L.\frac{\delta q}{\delta L} = n.q$ if the production function is homogeneous of degree n. Now if n > 1, more output than q will be required to make payments to the factors according to their marginal productivities. Thus, if there are increasing returns to scale (n > 1) and if the factors are paid according to their marginal productivities, then there will be a dificit in total output to pay those factors. On the other hand, if n < 1, i.e., if there are decreasing returns to scale, total output will not be fully utilised to pay the factors according to their marginal productivities. In that case, there will be a surplus of total output.

Let us cite some examples on product exhaustion theorem.

Example 2.22 : The production function is : $q = 2K^2 + 3L^2$. What will happen to total product if factors are paid according to their marginal productivities?

Solution : We have, $q = 2K^2 + 3L^2$

$$\therefore\ MP_K = \frac{\delta q}{\delta K} = 4K \text{ and } MP_L = \frac{\delta q}{\delta L} = 6L$$

Now, if factors are paid according to their marginal productivities, then total payment to capital and labour is given by= $K.\frac{\delta q}{\delta K}+L\frac{\partial q}{\delta L}$ = K.4K + L.6L

$= 4K^2 + 6L^2 = 2(2K^2 + 3L^2) = 2q.$

Thus, it shows that twice of total product would be necessary to pay the factors according to their marginal productivities. In other words, there would be a deficit in total output.

In fact, the given production function $q = 2K^2 + 3L^2$ is homogeneous of degree 2 (i.e., n = 2) or subject to IRS. So, as per Euler's theorem, amount of total factor payment would be = 2q (= nq) and hence there will be a deficit in total output.

Example 2.23 : The production function is $q = 30K^{\frac{1}{4}}L^{\frac{3}{4}}$. What will happen to total product if capital (K) and labour (L) are paid according to their marginal productivities?

Solution : We have, $q = 30K^{\frac{1}{4}}L^{\frac{3}{4}}$

Now, $MP_K = \frac{\delta q}{\delta K} = \frac{1}{4}\times 30K^{\frac{1}{4}-1}L^{\frac{3}{4}}$ and $MP_L = \frac{\delta q}{\delta K} = \frac{3}{4}\times 30K^{\frac{1}{4}}L^{\frac{3}{4}-1}$

So, $K.\frac{\delta q}{\delta K}+L.\frac{\delta q}{\delta L} = K.MP_K + L.MP_L$

$= K\times\frac{1}{4}\times 30K^{\frac{1}{4}-1}L^{\frac{3}{4}} + L\times\frac{3}{4}\times 30K^{\frac{1}{4}}L^{\frac{3}{4}-1} = \frac{1}{4}\times 30K^{\frac{1}{4}}L^{\frac{3}{4}} + \frac{3}{4}\times 30K^{\frac{1}{4}}L^{\frac{3}{4}}$

$= \frac{1}{4}\times q + \frac{3}{4}\times q = q\left(\frac{1}{4}+\frac{3}{4}\right) = q = TP.$

Thus, if factors are paid according to their marginal productivities, then total product will just be exhausted.

In fact, the given production function is homogeneous of degree one (please check it) or subject to constant returns to scale. Hence, as per Euler's theorem, total product will just be exhausted if factors are paid according to their marginal productivities.

Example 2.24 : The production function is $q = K^{\frac{1}{4}}L^{\frac{1}{4}}$. Will total product be just exhausted if factors are paid according to their marginal productivities?

Solution : We have, $q = K^{\frac{1}{4}}L^{\frac{1}{4}}$.

Now, $MP_K = \frac{\delta q}{\delta K} = \frac{1}{4}K^{\frac{1}{4}-1}.L^{\frac{1}{4}}$ and $MP_L = \frac{\delta q}{\delta L} = \frac{1}{4}.K^{\frac{1}{4}}L^{\frac{1}{4}-1}$

So, total payment to capital and labour = $K.\frac{\delta q}{\delta K}+L.\frac{\delta q}{\delta L}$

$$= K\times\frac{1}{4}K^{\frac{1}{4}-1}L^{\frac{1}{4}}+L\times\frac{1}{4}K^{\frac{1}{4}}L^{\frac{1}{4}-1} = \frac{1}{4}K^{\frac{1}{4}}L^{\frac{1}{4}}+\frac{1}{4}K^{\frac{1}{4}}L^{\frac{1}{4}}$$

$$= 2\times\frac{1}{4}K^{\frac{1}{4}}L^{\frac{1}{4}}=\frac{1}{2}\times K^{\frac{1}{4}}L^{\frac{1}{4}}=\frac{1}{2}.q$$

Thus, in this case, $\frac{1}{2}$ of total output is rqquired to make payments to factors as per their marginal productivities. So, there will be a surplus of total output.

Actually, in this case, the given production function is homogeneous of degree $\frac{1}{2}$ or subject to DRS $\left(n=\frac{1}{2}<1\right)$. Hence, as per Euler's theorem, total output required to make factor payment = nq = $\frac{1}{2}$ q. Thus there is surplus of total output.

2.10 Cobb-Douglas Production Function and its Properties

The Cobb-Douglas Production function is a particular functional form of the production function. It represents the relationship between two or more inputs–typically physical capital and labour–and the units of output that can be produced. It is based on the empirical study of the American manufacturing industry made by Charles W. Cobb and Paul H. Douglas. This function has some nice properties and hence is widely used in the analyses of economics and econometrics. The general form of the Cobb-Douglas production function is : $q = AK^{\alpha}L^{\beta}$ where A, α and β are positive parameters. Here q is output, K and L are inputs of capital and labour, respectively. The equation tells us that output (q) depends directly on K and L, and that part of output which cannot be explained by K and L is explained by A. Here A is the residual factor which stands for technical change.

Now, if we assume that $\alpha + \beta = 1$ so that $\beta = 1 - \alpha$, we can get a simpler form of the Cobb-Douglas production function. The function then takes the specific form :

$q = AK^{\alpha}L^{1-\alpha}$, $(0 < \alpha < 1)$

Taking this simple, specific form of the Cobb-Douglas production function, we shall now consider the major properties of this function.

Property 1 : There will be no output if both the inputs are not employed. That is, q = 0 if either K = 0 or L = 0. This means that both the inputs are necessary to have any output. There will be no output if only one input is used, no matter however large it is.

Property 2 : The simple and specific form of the Cobb-Douglas production function exhibits constant returns to scale (CRS) or it is homogeneous of degree 1.

Proof : We have the simple form of the Cobb-Douglas production function : $q = AK^{\alpha}L^{1-\alpha}$

Now, to examine its degree of homogeneity, we increase both K and L by λ times. The new value of output, say, $q^* = A(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = \lambda^{\alpha+1-\alpha}.\ AK^{\alpha}L^{1-\alpha} = \lambda^{1}.q = \lambda q$

Thus, the given function is homogeneous of degree 1. If we increase K and L by λ times, output also increases by λ times. Thus, the simple form of Cobb-Douglas production function exhibits CRS.

Property 3 : If the Cobb-Douglas production function is of the form, $q = AK^{\alpha}L^{1-\alpha}$ $(0 < \alpha < 1)$, then AP_K, AP_L, MP_K, MP_L will be diminishing or their slopes will be negative.

Proof : We have, $q = AK^{\alpha}L^{1-\alpha} (0 < \alpha < 1)$

Now to consider the slopes of AP_K, AP_L, MP_K and MP_L, we first derive their equations.

$$AP_K \equiv \frac{\text{Total output}}{K} = \frac{q}{K} = \frac{AK^{\alpha}L^{1-\alpha}}{K} = AK^{\alpha-1}L^{1-\alpha}$$

$$AP_L \equiv \frac{\text{Total output}}{L} = \frac{q}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L} = AK^{\alpha}L^{-\alpha}$$

$$MP_K \equiv \frac{\delta q}{\delta K} = \alpha AK^{\alpha-1} L^{1-\alpha}, \text{ and}$$

$$MP_L \equiv \frac{\partial q}{\partial L} = (1-\alpha)AK^{\alpha} L^{1-\alpha-1} = (1-\alpha)AK^{\alpha} L^{-\alpha}$$

Now, slope of $AP_K = \dfrac{\delta AP_K}{\delta K} = (\alpha - 1) AK^{\alpha-2} L^{1-\alpha} < 0$ as $0 < \alpha < 1$

Similarly, slope of $AP_L = \dfrac{\delta AP_L}{\delta L} = -\alpha AK^{\alpha} L^{-\alpha-1} < 0$ as $0 < \alpha < 1$

Now, slope of $MP_K = \dfrac{\delta MP_K}{\delta K} = \alpha(\alpha - 1)AK^{\alpha-2} L^{1-\alpha} < 0$ as $\alpha < 1$

and slope of $MP_L = \dfrac{\delta MP_L}{\delta L} = \alpha(1-\alpha)AK^{\alpha} L^{-\alpha-1} < 0$ as $0 < \alpha < 1$

Thus, if $q = AK^{\alpha}L^{1-\alpha}$, then AP_K, AP_L, MP_K, MP_L all are negatively sloped or they are diminishing.

Property 4 : Under Cobb-Douglas production function, marginal productivities of inputs will depend only on input ratio or will be homogeneous of degree zero.

Proof : We have, $q = AK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$

$$\text{Now, } MP_K \equiv \frac{\partial q}{\partial K} \text{ or } f_K = \alpha.AK^{\alpha-1}L^{1-\alpha} = \alpha.A\left(\frac{K}{L}\right)^{\alpha-1} = f_K\left(\frac{K}{L}\right).$$

$$\text{Similarly, } MP_L \equiv \frac{\partial q}{\partial L} \text{ or } f_L = (1-\alpha)AK^{\alpha}L^{-\alpha} = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = f_L\left(\frac{K}{L}\right)$$

Thus, both MP_K and MP_L depend on input ratio. Hence if we increase both K and L by a certian proportion the input-ratio $\frac{K}{L}$ will remain unchanged. Then MP_K and MP_L will remain unchanged. In other words, MP_K and MP_L are homogeneous of degree zero in K and L. Formally, we can show this. Suppose we increase K and L by λ times. Then the new value of MP_K, say, $MP_K^* = \alpha.A\left(\frac{\lambda K}{\lambda L}\right)^{\alpha-1} = \lambda^0.\alpha.A\left(\frac{K}{L}\right)^{\alpha-1} = \lambda^0.MP_K = MP_K$

Similarly, the new value of MP_L, say, MP_L^* is :

$$MP_L^* = (1-\alpha)A\left(\frac{\lambda K}{\lambda L}\right)^{\alpha} = \lambda^0(1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = \lambda^0.MP_L = MP_L$$

Thus, both MP_K and MP_L are homogeneous of degree zero. We know that if the production function is homogeneous of degree n, then the marginal productivities will be homogeneous of degree (n – 1). In our simple Cobb-Douglas form, the production function is homogeneous of degree 1 (n = 1), Naturally, the marginal productivities are homogeneous of degree zero (n – 1 = 1 – 1 = 0).

Property 5 : If the Cobb-Douglas production function is of the form $q = AK^{\alpha}L^{1-\alpha}$, then α and $(1-\alpha)$ represent elasticities of output with respect to K and L, respectively.

Proof : Elasticity of output (q) with respect to capital (K) is given by,

$$e_K = \frac{\frac{\partial q}{q}}{\frac{\partial K}{K}} = \frac{\partial q}{\partial K} \times \frac{K}{q} = \frac{K}{q}.\frac{\partial q}{\partial K}. \text{ Similarly, } e_L = \frac{\frac{\partial q}{q}}{\frac{\partial L}{L}} = \frac{\partial q}{\partial L} \times \frac{L}{q} = \frac{L}{q}.\frac{\partial q}{\partial L}.$$

Now, we have, $q = AK^{\alpha}L^{1-\alpha}$

$$\therefore \frac{\partial q}{\partial K} = \alpha.AK^{\alpha-1}L^{1-\alpha} = \frac{\alpha.AK^{\alpha}L^{1-\alpha}}{K} = \alpha.\frac{q}{K}$$

$$\therefore \frac{K}{q}\cdot\frac{\partial q}{\partial K} = \alpha \text{ i.e., } e_K = \alpha.$$

Similarly, $\frac{\partial q}{\partial L} = (1-\alpha)AK^{\alpha}L^{1-\alpha} = (1-\alpha).\frac{q}{L}$

$$\therefore \frac{L}{q}\cdot\frac{\delta q}{\partial L} = (1-\alpha) \text{ or, } e_L = (1-\alpha)$$

Thus, elasticity of output with respect to K is α and elasticity of output with respect to L is $(1-\alpha)$.

We can prove this in a slightly different manner.

We know, $e_K = \frac{K}{q}\frac{\partial q}{\partial K} = \frac{\frac{\partial q}{\partial K}}{\frac{q}{K}} = \frac{MP_K}{AP_K}$

Now, putting the values of MP_K and AP_K obtained in property 3, we get

$$e_K = \frac{MP_K}{AP_K} = \frac{\alpha.AK^{\alpha-1}L^{1-\alpha}}{AK^{\alpha-1}.L^{1-\alpha}} = \alpha$$

Similarly, $e_L = \frac{L}{q}\cdot\frac{\partial q}{\partial L} = \frac{\frac{\partial q}{\partial L}}{\frac{q}{L}} = \frac{MP_L}{AP_L}$

Now putting the values of MP_L and AP_L from property 3,

we get, $e_L = \frac{(1-\alpha)AK^{\alpha}L^{-\alpha}}{AK^{\alpha}L^{-\alpha}} = (1-\alpha).$

We can also prove our property by using log-definition of elasticity. We know that elasticity of q with respect to K,

$e_K = \frac{\partial \log q}{\delta \log K}$ and elasticity of q with respect to L, $e_L = \frac{\partial \log q}{\delta \log L}$

Now, we have, $q = AK^{\alpha}L^{1-\alpha}$

Taking log of both sides, we get,

$\log q = \log A + \alpha \log K + (1-\alpha)\log L$

Now, $e_K = \frac{\delta \log q}{\delta \log K} = 0 + \alpha.1 + 0 = \alpha$

Similarly, $e_L = \frac{\delta \log q}{\delta \log L} = 0 + 0 + (1 - \alpha).1 = (1 - \alpha)$

Property 6 : If $q = AK^{\alpha}L^{1-\alpha}$, then α and $(1 - \alpha)$ represent respective input shares if factors are paid according to their marginal productivities.

Proof : We have, $q = AK^{\alpha}L^{1-\alpha}$

$$\therefore \; MP_K = \frac{\delta q}{\delta K} = \alpha AK^{\alpha-1}L^{1-\alpha} = \alpha.\frac{AK^{\alpha}L^{1-\alpha}}{K} = \alpha.\frac{q}{K}$$

Similarly, $\frac{\delta q}{\delta L} = (1 - \alpha)AK^{\alpha}L^{1-\alpha-1} = (1 - \alpha)\frac{AK^{\alpha}L^{1-\alpha}}{L} = (1-\alpha)\frac{q}{L}$

We have been told that $P_K = MP_K$ and $P_L = MP_L$

Now, share of K in total output = $\frac{P_K.K}{q} = \frac{MP_K.K}{q} = \frac{\alpha.\frac{q}{K}.K}{q} = \alpha$

Similarly, share of L in total output = $\frac{P_L.L}{q} = \frac{(1-\alpha).\frac{q}{L}.L}{q} = (1-\alpha)$.

Thus, α and $(1 - \alpha)$ represent respective shares of capital and labour in total output.

Property 7 : Under Cobb-Douglas production function of the form $q = AK^{\alpha}L^{1-\alpha}$, total product will just be exhausted if factors are paid according to their marginal productivities. In other words, if $q = AK^{\alpha}L^{1-\alpha}$, then Euler's theorem will hold.

Proof : Our production function is : $q = AK^{\alpha}L^{1-\alpha}$.

Now, $MP_K \equiv \frac{\delta q}{\delta K} = \alpha.AK^{\alpha-1}.L^{1-\alpha} = \frac{\alpha.AK^{\alpha}L^{1-\alpha}}{K} = \alpha.\frac{q}{K}$.

Similarly, $MP_L \equiv \frac{\delta q}{\delta L} = (1 - \alpha).AK^{\alpha}L^{1-\alpha-1} = (1 - \alpha).\frac{\alpha.AK^{\alpha}L^{1-\alpha}}{L} = (1 - \alpha).\frac{q}{L}$

We are also given that $P_K = MP_K$ and $P_L = MP_L$.

Now, total payment to factors, K and L,

$= P_K.K + P_L.L = MP_K.K + MP_L.L = \alpha.\frac{q}{K}.K + (1 - \alpha).\frac{q}{L}.L$

$= \alpha q + (1 - \alpha)q = q(\alpha + 1 - \alpha) = q$ = Total output.

Thus, total output or total product (TP) will be exhausted if factors are paid according to their marginal productivities. This is known as product exhaustion theorem. This is also called Euler's theorem. The product exhaustion theorem is actually a corollary of Euler's theorem. In our earlier section (section 2.9.3) we have considered this corollary which states that if the function q = f(K, L) is homogeneous of degree 1, then $K.\frac{\partial q}{\partial k} + L.\frac{\partial q}{\partial L}$ = 1.q = q

Thus, the case of product exhaustion under Cobb-Douglas production function is just an application of the corollary of Euler's theorem. Hence, the product exhaustion theorem is loosely called Euler's theorem.

Property 8 : Under Cobb-Douglas production function, elasticity of substitution is equal to unity.

Proof : Elasticity of substitution, $\sigma = \dfrac{\frac{d(K/L)}{K/L}}{\frac{d(MRTS)}{MRTS}}$

or, $\sigma = \dfrac{\frac{d(K/L)}{K/L}}{\frac{d(MP_L/MP_K)}{MP_L/MP_K}} = \dfrac{d(K/L)}{K/L}.\dfrac{MP_L/MP_K}{d(MP_L MP_K)}$

Now, from our Cobb-Douglas production function $q = AK^{\alpha}L^{1-\alpha}$, we have obtained,

$MP_L = \frac{\partial q}{\partial L} = (1-\alpha).\frac{q}{L}$ and, $MP_K = \frac{\partial q}{\partial K} = \alpha.\frac{q}{K}$.

So, $\dfrac{MP_L}{MP_K} = \dfrac{(1-\alpha).q/L}{\alpha.q/K} = \dfrac{1-\alpha}{\alpha}.\dfrac{K}{L}$

$\therefore\ d\left(\dfrac{MP_L}{MP_K}\right) = \dfrac{1-\alpha}{\alpha}.d\left(\dfrac{K}{L}\right)$

Putting these values in the expression of elasticity of substitution, we get,

$$\sigma = \frac{d(K/L)}{K/L}\frac{MP_L/MP_K}{d(MP_L/MP_K)} = \frac{d\left(\frac{K}{L}\right)}{\frac{K}{L}}.\frac{\frac{1-\alpha}{\alpha}.\frac{K}{L}}{\frac{1-\alpha}{\alpha}.d\left(\frac{K}{L}\right)} = 1 \text{ (proved)}$$

Alternative proof : In terms of log-definition, the elasticity of substitution is given by, $\sigma = \dfrac{d\log\left(\dfrac{K}{L}\right)}{d\log\left(\dfrac{MP_L}{MP_K}\right)}$

Now, $\dfrac{MP_L}{MP_K} = \dfrac{(1-\alpha).q/L}{\alpha.q/K} = \dfrac{1-\alpha}{\alpha}.\dfrac{K}{L}$

Taking log of both sides, we get,

$$\log\left(\frac{MP_L}{MP_K}\right) = \log\left(\frac{1-\alpha}{\alpha}\right) + \log\left(\frac{K}{L}\right)$$

or, $\log\left(\dfrac{K}{L}\right) = \log\left(\dfrac{MP_L}{MP_K}\right) - \log\left(\dfrac{1-\alpha}{\alpha}\right)$

Now, elasticity of substitution, $\sigma = \dfrac{d\log(K/L)}{d\log(MP_L/MP_K)}$ = 1 – 0 = 1 (proved)

Property 9 : Under Cobb-Douglas production function, the expansion path is a straight line passing through the origin, provided input prices are fixed.

Proof : An expansion path of a firm is the locus of successive tangency points between the isoquants and the parallel iso-cost lines. Hence, at each point on an expansion path, slope of isoquant = slope of iso-cost line i.e., $-\dfrac{MP_L}{MP_K} = -\dfrac{P_L}{P_K}$ or, $\dfrac{MP_L}{MP_K} = \dfrac{P_L}{P_K}$.

This is the equation of an expansion path. Now, when $q = AK^{\alpha}L^{1-\alpha}$, we have,

$MP_K = \dfrac{\delta q}{\delta K} = \alpha.\dfrac{q}{K}$ and $MP_L = \dfrac{\partial q}{\partial L} = (1-\alpha)\dfrac{q}{L}$.

Putting these values of MP_K and MP_L in the equation of expansion path, we get,

$\dfrac{(1-\alpha).\dfrac{q}{L}}{\alpha.\dfrac{q}{K}} = \dfrac{P_L}{P_K}$, or $\dfrac{1-\alpha}{\alpha}.\dfrac{K}{L} = \dfrac{P_L}{P_K}$ or, $K = \dfrac{\alpha}{1-\alpha}.\dfrac{P_L}{P_K}.L$.

This is an equation of a straight line passing through the origin. Hence, under Cobb-Douglas production function, the expansion path will be a straight line passing through the origin.

Property 10 : Under Cobb-Douglas production function, isoquants will be downward sloping and convex to the origin.

Solution : We have the Cobb-Douglas production function, $q = AK^{\alpha}L^{1-\alpha}$. This function has 3 variables : q, K and L. So, to plot this function, we require a three-dimenstinal diagram. To avoid it, we assume q as fixed at a certain value, say, q_0. This q_0 amount of output may be produced by different combinations of K and L. The locus of all such combinations of K and L which can produce a certain q_0 level of output form an isoquant or an equal product curve. So, the equation of an iso-quant under Cobb-Douglas production function is : $q_0 = AK^{\alpha}L^{1-\alpha}$. It now involves two variables : K and L. We may plot it on a two dimensional diagram measuring K along the vertical axis and L along the horizontal axis. In other words, we may plot it as K = f(L). So, we express the isoquant $q_0 = AK^{\alpha}L^{1-\alpha}$ as K = f(L).

We have, $AK^{\alpha}L^{1-\alpha} = q_0$

or, $K^{\alpha} = \left(\frac{q_0}{A}\right)L^{\alpha-1}$, or, $K^{\alpha} = \left(\frac{q_0}{A}\right)^{\frac{1}{\alpha}}.L^{\frac{\alpha-1}{\alpha}}$, $(0 < \alpha < 1)$

Now, slope of the iso-quant = $\frac{dK}{dL} = \left(\frac{\alpha-1}{\alpha}\right)\left(\frac{q_0}{A}\right)^{\frac{1}{\alpha}} \cdot L^{\frac{\alpha-1}{\alpha}-1} < 0.$

Here $\frac{dK}{dL} < 0$ as $\alpha < 1$. Thus an isoquant under Cobb-Douglas production function will be negatively sloped. To know its curvature, we have to differentiate $\frac{dK}{dL}$ further with respect to L. That will give us the change in slope of the isoquant.

The change in slope = $\frac{d}{dL}\left(\frac{dK}{dL}\right) = \frac{d^2K}{dL^2} = \left(\frac{\alpha-1}{\alpha}-1\right)\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{q_0}{A}\right)^{\frac{1}{\alpha}} \cdot L^{\frac{\alpha-1}{\alpha}-1-1}$

$$= -\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{q_0}{A}\right)^{\frac{1}{\alpha}}.L^{\frac{\alpha-1}{\alpha}-2} = \frac{1}{\alpha}.\frac{1-\alpha}{\alpha}.\left(\frac{q_0}{A}\right)^{\frac{1}{\alpha}}.L^{\frac{\alpha-1}{\alpha}-2} > 0 \text{ as } 0 < \alpha < 1$$

This implies that the slope of the isoquant rises. But its slope was originally negative. So, it implies that the absolute slope of the isoquant falls. This will happen if the isoquant is convex to the origin. Thus, under Cobb-Douglas production function, the isoquant will be convex to the origin (proved)

The general form of the Cobb-Douglas production function as we have mentioned, is : $q = AK^{\alpha}L^{\beta}$, $\alpha + \beta \gtreqless 1$. Its major properties are mentioned below. We are not deducing the proofs of those properties; they are being left to the students as an exercise.

Property 1 : The Cobb-Douglas production function of the form $q = AK^{\alpha}L^{\beta}$ is homogeneous of degree $(\alpha + \beta)$. If $\alpha + \beta > 1$, it will display IRS. If $\alpha + \beta = 1$, it will show CRS. If $\alpha + \beta < 1$, it will imply DRS. Thus, the general form of the Cobb-Douglas production function can exhibit all three types of returns to scale.

Property 2 : Under Cobb-Douglas production function $q = AK^{\alpha}L^{\beta}$, α and β represent elasticities of output with respect to capital(K) and labour(L), respectively.

Property 3 : In the production function $q = AK^{\alpha}L^{\beta}$, α and β represent the share of capital and labour in total output, respectively, if factors are paid according to their marginal productivities.

Property 4 : If the Cobb-Douglas production function is of the form $q = AK^{\alpha}L^{\beta}$, then marginal productivities will be homogeneous of degree $(\alpha + \beta - 1)$.

Property 5 : Under Cobb-Douglas production function $q = AK^{\alpha}L^{\beta}$, the elasticity of factor substitution is unity.

Property 6 : The Cobb-Douglas production function $q = AK^{\alpha}L^{\beta}$ can be represented by downward sloping convex isoquants.

Property 7 : Under Cobb-Douglas production function $q = AK^{\alpha}L^{\beta}$, the expansion path will be a straight line passing through the origin, provided input prices are fixed. Let us consider some examples related to the Cobb-Douglas production function.

Example 2.25 : The production function is : $q = 80K^{\frac{1}{5}}L^{\frac{4}{5}}$. What will be the shapes of AP_K, MP_K, AP_L and MP_L curves?

Solution : We have, $q = 80K^{\frac{1}{5}}L^{\frac{4}{5}}$

Now, $AP_K = \frac{q}{K} = \frac{80K^{\frac{1}{5}}L^{\frac{4}{5}}}{K} = 80K^{-\frac{4}{5}}L^{\frac{4}{5}}$, and

$$MP_K = \frac{\delta q}{\delta K} = \frac{1}{5}80K^{\frac{1}{5}-1}L^{\frac{4}{5}} = 16K^{-\frac{4}{5}}L^{\frac{4}{5}}$$

Now, slope of $AP_K = \frac{\delta AP_K}{\delta K} = -\frac{4}{5}\times 80.K^{-\frac{4}{5}-1}.L^{\frac{4}{5}} = -64K^{-\frac{9}{5}}L^{\frac{4}{5}} < 0$

Similarly, slope of $MP_K = \frac{\delta}{\delta K}(MP_K) = \frac{\delta}{\partial K}\left(\frac{\delta q}{\delta K}\right) = \frac{\delta^2 q}{\delta K^2}$

$$= -\frac{4}{5}\times 16.K^{-\frac{4}{5}-1}.L^{\frac{4}{5}} = -\frac{64}{5}K^{-\frac{9}{5}}L^{\frac{4}{5}} < 0$$

We see that slopes of AP_K and MP_K curves are negative. Thus, AP_K and MP_K are diminishing, if more of K is used. Similarly we can show that AP_L and MP_L will be diminishing if more of L is used.

Example 2.16 : The production function is $q = 30K^2L^3$. Derive the expansion path of the firm if $P_K = 10$ and $P_L = 20$.

Solution : Along an expansion path, $\frac{MP_L}{MP_K} = \frac{P_L}{P_K}$.

Now, we have, $q = 30K^2L^3$

$$MP_K = \frac{\delta q}{\partial K} = 2\times 30K^{2-1}L^3 = 2\times 30KL^3$$

$$MP_L = \frac{\delta q}{\delta L} = 2\times 30K^2L^{3-1} = 3\times 30K^2L^2$$

Further, we are given that $P_L = 20$ and $P_K = 10$

Putting, these values we get the equation of the expansion path.

$$\frac{MP_L}{MP_K} = \frac{P_L}{P_K} \quad \text{or,} \quad \frac{3\times 30.K^2L^2}{2\times 30.KL^3} = \frac{20}{10} \quad \text{or,} \quad \frac{3}{2}.\frac{K}{L} = 2$$

or, $K = \frac{2\times 2}{3}.L$ or, $K = \frac{4}{3}L$. This is our desired expansion path which is here a straight line passing through the origin.

Example 2.17 : Let the production function be : $Y = 12K^{\frac{1}{4}}L^{\frac{3}{4}}$

Calculate the elasticity of substitution.

Solution : The elasticity of substitution is given by the formula,

$$\sigma = \frac{d(K/L)}{K/L}\Bigg/\frac{d(MP_L/MP_K)}{MP_L/MP_K} = \frac{d(K/L)}{K/L}\times\frac{MP_L/MP_K}{d(MP_L/MP_K)}$$

Now, we have, $Y = 12K^{\frac{1}{4}}L^{\frac{3}{4}}$

$$MP_K = \frac{\delta Y}{\delta K} = \frac{1}{4}\times 12.K^{\frac{1}{4}-1}L^{\frac{3}{4}} = 3.K^{-\frac{3}{4}}L^{\frac{3}{4}}$$

$$MP_L = \frac{\partial Y}{\partial L} = \frac{3}{4}\times 12.K^{\frac{1}{4}}L^{\frac{3}{4}-1} = 9.K^{\frac{1}{4}}L^{-\frac{1}{4}} = 3.K^{-\frac{3}{4}}L^{\frac{3}{4}}$$

So, $\frac{MP_L}{MP_K} = \frac{9.K^{\frac{1}{4}}L^{-\frac{1}{4}}}{3.K^{-\frac{3}{4}}L^{\frac{3}{4}}} = 3.\frac{K^{\frac{1}{4}+\frac{3}{4}}}{L^{\frac{3}{4}+\frac{1}{4}}} = 3.\frac{K}{L}$ $\quad\therefore\ d\left(\frac{MP_L}{MP_K}\right) = 3.d\left(\frac{K}{L}\right)$.

Putting the values of $\frac{MP_L}{MP_K}$ and $d\left(\frac{MP_L}{MP_K}\right)$ in the formula of σ, we get,

$$\sigma = \frac{d(K/L)}{K/L} \times \frac{3.(K/L)}{3.d(K/L)} = 1$$ **(Ans.)**

Alternative method : The formula of elasticity of substitution can be written as,

$$\sigma = \frac{d\log(K/L)}{d\log(MP_L/MP_K)}$$

Now, we have got, $\frac{MP_L}{MP_K} = 3\frac{K}{L}$

Taking log of both sides, we get, $\log\left(\frac{MP_L}{MP_K}\right) = \log 3 + \log\left(\frac{K}{L}\right)$

or, $\log\left(\frac{K}{L}\right) = \log\left(\frac{MP_L}{MP_K}\right) - \log 3$

Now, elasticity of substitution (σ) = $\frac{d\log(K/L)}{d\log(MP_L/MP_K)} = 1 - 0 = 1$ **(Ans.)**

2.11 CES Production Function and its Properties

The CES production function is a neoclassical production function that displays constant elasticity of substitution. In other words, the production function or the production technology has a constant percentage change in factor (e.g., capital and labour) proportions due to a percentage change in marginal rate of technical substitution (MRTS). This function has been developed by Arrow, Chenery, Minhas and Solow in a celebrated paper in 1961. The formal equation of the CES (constant elasticity of substitution) production function is :

$q \quad A \quad K \quad (1 \quad)L^{\,\frac{1}{\;}}$ where q = output,

K = capital, L = Labour ($A > 0$, $0 < \alpha < 1$, $0 \neq \rho > -1$). This function has three parameters : A, α and ρ. A indicates the state of technology and organisational aspetcs of production. Hence A is called technological parameter. α determines the relative factor shares in the total output and so α is called the distribution parameter. The value of ρ determines the elasticity of substitution between inputs. Hence ρ is called the factor substitution parameter.

The CES production function has some important properties. We consider its major properties one by one :

Property 1 : The CES production function is homogeneous of degree one or exhibits constant returns to scale.

Proof : We have the CES production function, $q \quad A \quad K \quad (1 \quad)L^{\,\frac{1}{\;}}$.

Now we increase both K and L by λ times in order to see the degree of homogeneity of this function.

The new output level, say, q* is then

$$q^* = A \quad (\;K) \quad (1 \quad)(\;L)^{\,\frac{1}{\;}}$$

$$= \lambda^{-\rho\times-\frac{1}{\rho}}.A\left[\lambda K^{-\rho}+(1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}} = \lambda^1.q = \lambda q$$

So the given CES production function is homogeneous of degree one. Hence the function displays constant returns to scale (CRS). We see that if we increase both K and L by λ times, output (q) also changes by λ times.

Property 2 : Under CES production function, MP_K and MP_L are homogeneous of degree zero.

Proof : We have, $q = A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}$

$$\text{Now, } MP_K = \frac{\delta q}{\delta K} \quad \text{or, } f_K = -\frac{1}{\rho}.A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}-1}.\alpha(-\rho)K^{-\rho-1}$$

$$= \alpha.A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{(1+\rho)}{\rho}}.K^{-(1+\rho)}$$

$$\text{Similarly, } MP_L = \frac{\delta q}{\partial L} \equiv f_L = -\frac{1}{\rho}A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}-1}(1-\alpha)(-\rho).L^{-(1+\rho)}$$

$$= (1-\alpha).\frac{A^{\rho+1}}{A^{\rho}}.\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{(1+\rho)}{\rho}}.L^{-(1+\rho)}.$$

In order to determine the degree of homogeneity of MP_K and MP_L we increase both K and L by λ times. The new MP_K, say,

$$MP_K^* = \alpha A\left[\alpha(\lambda K)^{-\rho} + (1-\alpha)(\lambda L)^{-\rho}\right]^{-\frac{(1+\rho)}{\rho}}.(\lambda K)^{-(1+\rho)}$$

$$= \lambda^{(1+\rho)}.\lambda^{-(1+\rho)}.\alpha A\left[\alpha(\lambda K)^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1+\rho}{\rho}} K^{-(1+\rho)} = \lambda^0.MP_K = MP_K$$

Thus, MP_K is homogeneous of degree zero.

Similarly, the new MP_L, say,

$$MP_L^* \quad (1 \quad)A \quad (\; K) \quad (1 \quad)(\; L) \quad \frac{(1 \quad)}{} .(\; L)^{(1 \quad)}$$

$$= \lambda^{(1+\rho)}.\lambda^{-(1+\rho)}.(1-\alpha)A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{(1+\rho)}{\rho}}.L^{-(1+\rho)} = \lambda^0.MP_L = MP_L.$$

Thus, MP_L is also homogeneous of degree zero. If both K and L are changed by λ proportion, their marginal productivities remain unaffected.

Property 3 : Under CES production function, if the factors are paid according to their marginal productivities, then total product will be exhausted. In other words, product exhaustion theorem or Euler's theorem will hold under CES production function.

Proof : Here we have to prove that $K.\frac{\delta q}{\partial K} + L.\frac{\partial q}{\delta L} = q$ i.e., $K.MP_K + L.MP_L = q$ or, using a different notation, we have to prove,

$Kf_K + L.f_L = q$ ...(1)

Now, we have, $q = \left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}$.

Now, $MP_K \equiv \frac{\delta q}{\delta K} \equiv f_K = -\frac{1}{\rho}A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}-1}\alpha.(1-\rho)K^{-1-\rho}$

This can be written as, $f_K = \alpha.\frac{A^{1+\rho}}{A^{\rho}}\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{\frac{-(1+\rho)}{\rho}} K^{-(1+\rho)}$

$$= \frac{\alpha}{A^{\rho}}.\frac{q^{1+\rho}}{K^{1+\rho}} = \frac{\alpha}{A^{\rho}}.\left(\frac{q}{K}\right)^{1+\rho}.$$

Similarly, $MP_L \quad \frac{q}{L} \quad f_L \quad \frac{1}{} A \quad K \quad (1 \quad)L \quad ^{\frac{1}{} \; 1}(1 \quad)(\quad)L \quad ^{1}$

$$= (1-\alpha).\frac{A^{\rho+1}}{A^{\rho}}.\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{(1+\rho)}{\rho}}.L^{-(1+\rho)}$$

$$= \frac{1-\alpha}{A^{\rho}}.\frac{q^{1+\rho}}{L^{1+\rho}} = \frac{(1-\alpha)}{A^{\rho}}.\left(\frac{q}{L}\right)^{1+\rho}.$$

Now, putting the values of MP_K and MP_L in the LHS of equation(1),

we get, $K.f_K + Lf_L = K.\frac{\alpha}{A^{\rho}}.\left(\frac{q}{K}\right)^{1+\rho} + L.\frac{(1-\alpha)}{A^{\rho}}.\left(\frac{q}{L}\right)^{1+\rho}$

$$= \frac{q^{1+\rho}}{A^{\rho}}\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]$$

$$= q^{1+\rho}.A^{-\rho}\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}(-\rho)}$$

$= q^{1+\rho}.q^{-\rho} = q^{1+\rho-\rho} = q =$ R.H.S of equation (1)

Atternatively, we have,

$$K.f_K + L.f_L = \frac{q^{1+\rho}}{A^{\rho}}\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]$$

Now, our production function is :

$$q = A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}$$

or, $\frac{q}{A} = \left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}$

$$\therefore \left(\frac{q}{A}\right)^{-\rho} = \alpha K^{-\rho} + (1-\alpha)L^{-\rho}.$$

Putting this value we get,

$K.f_K + L.f_L = \frac{q^{1+\rho}}{A^{\rho}}.\left(\frac{q}{A}\right)^{-\rho} = \frac{q^{1+\rho-\rho}}{A^{\rho-\rho}} = q$ = Total product (Proved)

Thus, we see that if factors are paid according to their marginal productivities under CES production function, then total payment to the factors = $K.f_K + L.f_L = q$. In other words, total product (= q) is just exhausted. In other words, Euler's theorem or product exhaustion theorem holds under CES.

Property 4 : Under CES production function, marginal productivities of inputs are positive but diminishing.

Proof : We have, $q = A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}$.

Now, $MP_K \equiv \frac{\delta q}{\delta K} = f_K = -\frac{1}{\rho} A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}-1}.\alpha.(-\rho).K^{-\rho-1}$

$$= \frac{\alpha}{A^{\rho}}.A^{(1+\rho)}.\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{\frac{-(1+\rho)}{\rho}}.K^{-(1+\rho)}$$

$$= \frac{\alpha}{A^{\rho}}.\frac{q^{1+\rho}}{K^{1+\rho}} = \frac{\alpha}{A^{\rho}}\left(\frac{q}{K}\right)^{1+\rho} > 0$$

Similarly, $MP_L \equiv \frac{\delta q}{\delta L} = f_L = -\frac{1}{\rho} A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}-1}(1-\alpha)(-\rho).L^{-\rho-1}$

$$= \frac{1-\alpha}{A^{\rho}}.A^{(1+\rho)}.\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{\frac{-(1+\rho)}{\rho}}.L^{-(1+\rho)}$$

$$= \frac{1-\alpha}{A^{\rho}}.\frac{q^{(1+\rho)}}{L^{1+\rho}} = \frac{1-\alpha}{A^{\rho}}.\left(\frac{q}{L}\right)^{1+\rho} > 0$$

Now, slope of $MP_K = \frac{\delta}{\delta K}\left(\frac{\delta q}{\delta K}\right) = f_{KK} = \frac{\delta^2 q}{\delta K^2}$

$$= \frac{\alpha}{A^{\rho}}(1+\rho)\left(\frac{q}{K}\right)^{\rho}\left[\frac{f_K.K - q}{K^2}\right]$$

Now, from Euler's theorem we know that $f_K.K + f_L.L = q$

So, $f_K.K - q = -f_L.L < 0$ as $f_L > 0$, So, slope of MP_K i.e., f_{KK} under CES production function < 0, i.e., MP_K or f_K is diminishing.

Similarly, slope of $MP_L = \frac{\delta}{\delta L}\left(\frac{\delta q}{\delta L}\right) = f_{LL} = \frac{\delta^2 q}{\delta L^2} = \frac{1-\alpha}{A^{\rho}}(1+\rho)\left(\frac{q}{L}\right)\left[\frac{f_L.L - q}{L^2}\right]$

Again, from Euler's theorem, we have, $f_K K + f_L L = q$

or, $f_L L - q = -f_K K < 0$ as $f_K > 0$.

So, slope of MP_L i.e., f_{LL} under CES production function < 0. i.e., MP_L or f_L is diminishing.

Thus, under CES production function, MP_K and MP_L are positive but diminishing.

Property 5 : Under CES production function, the expansion path of the firm is a straight line passing through the origin, provided input prices are fixed.

Proof : We know that along an expansion path, slope of isoquant = slope of iso-cost line, i.e, $-\dfrac{MP_L}{MP_K} = -\dfrac{P_L}{P_K}$ or $\dfrac{MP_L}{MP_K} = \dfrac{P_L}{P_K}$.

Now, under CES production function, we have got,

$$MP_L = \frac{1-\alpha}{A^{\rho}}.\left(\frac{q}{L}\right)^{1+\rho} \text{ and } MP_K = \frac{\alpha}{A^{\rho}}.\left(\frac{q}{K}\right)^{1+\rho}$$

Putting these values we get, $\dfrac{\dfrac{1-\alpha}{A^{\rho}}.\left(\dfrac{q}{L}\right)^{1+\rho}}{\dfrac{\alpha}{A^{\rho}}.\left(\dfrac{q}{K}\right)^{1+\rho}} = \dfrac{\rho_L}{P_K}$ or, $\dfrac{1-\alpha}{\alpha}\left(\dfrac{K}{L}\right)^{1+\rho} = \dfrac{P_L}{P_K}$

or, $\left(\dfrac{K}{L}\right)^{1+\rho} = \dfrac{\alpha}{1-\alpha}.\dfrac{P_L}{P_K}$ or, $\dfrac{K}{L} = \left(\dfrac{\alpha}{1-\alpha}.\dfrac{P_L}{P_K}\right)^{\frac{1}{1+\rho}}$ $\therefore K = \left(\dfrac{\alpha}{1-\alpha}.\dfrac{P_L}{P_K}\right)^{\frac{1}{1+\rho}}.L$

This is the equation of the expansion path under CES production function. Clearly, this is a straight line passing through the origin. Hence, under CES production function, the expansion path will be a straight line passing through the origin.

Property 6 : Under CES production function, isoquants will be downward sloping and convex to the origin.

Proof : We know that if the production function is : q = f(K, L) then the equation of a particular isoquant representing a particular level of output (say, q_0) is, q_0 = f(K, L).

Taking total derivative of this function, we get, $dq_0 = \dfrac{\partial q}{\delta K}.dK + \dfrac{\partial q}{\partial L}.dL$

= MP_K.dK + MP_L.dL

Or, using a different notation, $dq_0 = f_K dK + f_L dL$.

Now, along a particular isoquant, q_0, the level of output is fixed i.e., $dq_0 = 0$. So, we have, f_K.dk + f_LdL = 0. If K is plotted vertically and L is plotted horizontally while drawing an isoquant, then the slope of the isoquant is $\dfrac{dK}{dL}$. Thus, we have,

f_K.dK + f_L.dL = 0 or, f_K.dK = –f_L.dL

So, $\dfrac{dK}{dL}$ or slope of the isoquant = $-\dfrac{f_L}{f_K} = -\dfrac{MP_L}{MP_K}$

Now, in the context of CES production function, we have obtained,

$$MP_L = f_L = \frac{1-\alpha}{A^\rho}\left(\frac{q}{L}\right)^{1+\rho} \text{ and } MP_K = f_K = \frac{\alpha}{A^\rho}\left(\frac{q}{K}\right)^{1+\rho}$$

So, $$\frac{dK}{dL} = -\frac{f_L}{f_K} = -\frac{1-\alpha}{A^\rho}.\left(\frac{q}{L}\right)^{1+\rho} \Big/ \frac{\alpha}{A^\rho}\left(\frac{q}{K}\right)^{1+\rho} = -\frac{1-\alpha}{\alpha}.\left(\frac{K}{L}\right)^{1+\rho} < 0$$

Thus, slope of an isoquant obtained from the CES production function is negative i.e., the isoquant will be downward sloping.

To know the curvature of the isoquant we have to know the change in slope due to change in L, i.e., we have to know the sign of $\frac{d}{dL}\left(\frac{dK}{dL}\right)$ or $\frac{d^2K}{dL^2}$.

So, change in slope = $\frac{d^2K}{dL^2} = \frac{d}{dL}\left(\frac{dK}{dL}\right)$.

$$= -\frac{1-\alpha}{\alpha}.(1+\rho)\left(\frac{K}{L}\right)^\rho \left[\frac{\frac{dK}{dL}.L - K.1}{L^2}\right]$$

$$= -\frac{1-\alpha}{\alpha}(1+\rho)\left(\frac{K}{L}\right)^\rho \left[\frac{-\frac{(1-\alpha)}{\alpha}.\left(\frac{K}{L}\right)^{1+\rho} L - K}{L^2}\right]$$

$$= \frac{1-\alpha}{\alpha}(1+\rho)\left(\frac{K}{L}\right)^\rho \left[\frac{\left(\frac{(1-\alpha)}{\alpha}\right)\left(\frac{K}{L}\right)^{1+\rho} L + K}{L^2}\right] > 0$$

This implies that slope of the isoquant will increase or the absolute slope (= MRTS) will be diminishing. This again implies that the isoquant obtained from the CES production function will be convex to the origin.

Property 7 : Under CES production function, the elasticity of factor substitution is a constant and is given by $\frac{1}{1+\rho}$. [That is why the function has been named as CES or constant elasticity of substitution production function]

Proof : We know that elasticity of factor substituion, say,

$$\sigma = \frac{d\log(K/L)}{d\log(MRTS)} = \frac{d\log(K/L)}{d\log(MP_L/MP_K)}.$$

Now, under CES production function we have,

$$MP_L = \frac{1-\alpha}{A^\rho}\left(\frac{q}{L}\right)^{1+\rho} \text{ and } MP_K = \frac{\alpha}{A^\rho}\left(\frac{q}{K}\right)^{1+\rho}$$

$$\text{Hence, } \frac{MP_L}{MP_K} = \frac{\frac{1-\alpha}{A^\rho}\left(\frac{q}{L}\right)^{1+\rho}}{\frac{\alpha}{A^\rho}\left(\frac{q}{K}\right)^{1+\rho}} = \frac{1-\alpha}{\alpha}.\left(\frac{K}{L}\right)^{1+\rho}$$

Now taking log of both sides, we have,

$$\log\left(\frac{MP_L}{MP_K}\right) = \log\frac{1-\alpha}{\alpha} + (1+\rho)\log\left(\frac{K}{L}\right)$$

$$\text{or, } (1+\rho)\log\left(\frac{K}{L}\right) = \log\left(\frac{MP_L}{MP_K}\right) - \log\left(\frac{1-\alpha}{\alpha}\right)$$

$$\therefore \log\left(\frac{K}{L}\right) = \frac{1}{1+\rho}.\log\left(\frac{MP_L}{MP_K}\right) - \frac{1}{1+\rho}.\log\left(\frac{1-\alpha}{\alpha}\right)$$

Now, elasticity of factor substitution,

$$\sigma = \frac{d\log\left(\frac{K}{L}\right)}{d\log\left(\frac{MP_L}{MP_K}\right)} = \frac{1}{1+\rho}.1 - 0 = \frac{1}{1+\rho}$$

Thus, the elasticity of factor substitution under CES production function is a constant and is equal to $\frac{1}{1+\rho}$.

Its magnitude will depend on the value of the parameter ρ as follows :

(i) If $-1 < \rho < 1$, then $\sigma > 1$

(ii) If $\rho = 0$, the $\sigma = 1$ i.e., elasticity of factor substitution = 1. This happens under Cobb-Douglas production function where $\sigma = 1$.

(iii) If 0 < σ < ∝, the σ < 1.

In fact, CES production function is a general case of Cobb-Douglas type production function. All the major properties of Cobb-Douglas production function holds in the case of CES production function except in the case of elasticity of factor substitution. In the case of Cobb-Douglas production function, the elasticity of factor substitution = 1. while in the case of CES production function, the elasticity of factor substitution = $\frac{1}{1+\rho}$. We may note that this value will tend to 1 as $\rho \to \propto$. Thus, Cobb-Douglas production function is a special case of CES production function when the parameter $\rho \to \propto$.

Let us consider some examples on different properties of CES and Cobb-Douglas type production functions.

Example 2.28 : Examine whether product exhaustion theorem will hold if factors are paid according to their marginal productivities and the production function is : $q = \frac{a}{L} \quad \frac{b}{K} \quad ^{\frac{1}{\ }}$. Also determine the elasticity of factor substitution.

Solution : We have the production function, $q = \frac{a}{L} \quad \frac{b}{K} \quad ^{\frac{1}{\ }}$. It can be rewritten as :

$q = aL \quad bK \quad ^{\frac{1}{\ }}$. Thus the given production function is of standard CES form. So in this case, Euler's theorem or product exhaustion theorem will hold. Further, in this case, elasticity of factor substitution will be equal to unity. (Prove yourself)

Example 2.29 : Production function is given as $q = 10 - \frac{1}{K} - \frac{1}{L}$. Determine elasticity of factor substitution(σ).

Solution : We have, $q = 10 - \frac{1}{K} - \frac{1}{L}$. We know that elasticity of factor substitution,

$$\sigma = \frac{d\log\left(\frac{K}{L}\right)}{d\log\left(\frac{MP_L}{MP_K}\right)}.$$

Now, $MP_L = \frac{\delta q}{\delta L} = \frac{1}{L^2}$ and $MP_K = \frac{\partial q}{\delta K} = \frac{1}{K^2}$

$$\therefore \frac{MP_L}{MP_K} = \frac{\frac{1}{L^2}}{\frac{1}{K^2}} = \frac{K^2}{L^2} = \left(\frac{K}{L}\right)^2$$

Taking log of both sides, we get,

$$\log\left(\frac{MP_L}{MP_K}\right) = 2\log\left(\frac{K}{L}\right) \quad \text{or, } \log\left(\frac{K}{L}\right) = \frac{1}{2}.\log\left(\frac{MP_L}{MP_K}\right).$$

Now, elasticity of factor substitution = $\sigma = \frac{d\log\left(\frac{K}{L}\right)}{d\log\left(\frac{MP_L}{MP_K}\right)} = \frac{1}{2}.$

So, elasticity of factor substitution in this case in equal to half.

Example 2.30 : Will product exahaustion theorem hold if $q = \sqrt{\alpha L^2 + \beta K^2}$ and factors are paid according to their marginal productivities?

Solution : We have, q = $\left(\alpha L^2 + \beta K^2\right)^{\frac{1}{2}}$.

Now, $MP_L = \frac{\partial q}{\partial L} = \frac{\frac{1}{2}(\alpha L^2 + \beta K^2)^{\frac{1}{2}}(2\alpha).L}{\alpha L^2 + \beta K^2} = \frac{\alpha.qL}{\alpha L^2 + \beta K^2}$

Similarly, $MP_K = \frac{\delta q}{\delta K} = \frac{\beta qK}{\alpha L^2 + \beta K^2}$

Now, total payment made to L and K is = $L.MP_L + K.MP_K$

$$= \frac{\alpha qL^2}{\alpha L^2 + \beta K^2} + \frac{\beta qL^2}{\alpha L^2 + \beta K^2} = \frac{\alpha qL^2 + \beta qK^2}{\alpha L^2 + \beta K^2} = \frac{q(\alpha L^2 + \beta K^2)}{(\alpha L^2 + \beta K^2)} = q = TP.$$

So, total product is exhausted if the factors are paid according to their marginal productivities when the production function is $q = \sqrt{\alpha L^2 + \beta K^2}$. In fact, here the given production function is homogeneous of degree 1 (please check) and hence total product exhausts.

2.12 Summary

1. AVERAGE AND MARGINAL FUNCTIONS AND THEIR USES

In Economics, we very often use the concept of function. For example, we have demand function, supply function, production function, cost function, revenue function, profit function, consumption function, saving function, investment function and so on. If y = f(x), then $\frac{y}{x}$ is called the average function. That is, average function gives us value of the dependent variable (y) per unit of the independent variable(x). On the other hand, marginal function is the first order derivative of the function y = f(x) i.e., marginal function of y = f(x) is $\frac{dy}{dx}$ or $f'(x)$. It gives us the change in the value of dependent variable due to one unit change in the independent variable. There are various uses of average and marginal functions in Economics. In particular, concepts of average and marginal functions may be used to know the elasticity of dependent variable with respect to its independent variable. For example, using the concepts of marginal and average functions we can know the price elasticity of demand, income elasticity of demand, cross (price) elasticity of demand, elasticity of cost and so on.

2. MAJOR APPLICATIONS OF DERIVATIVES IN ECONOMICS

If y = f(x), then its first derivative is given by $\frac{dy}{dx}$ or $f'(x)$. This derivative has so many uses in Economics. It is used to determine different types of elasticities. More importantly, the derivative helps us to know the marginal value of a variable which is so important in economic decision-making like profit maximisation, cost minimisation, etc. Further, derivative helps us to know the slope and curvature of a function.

3. RELATION BETWEEN PRICE ELASTICITY OF DEMAND AND TOTAL EXPENDITURE OR TOTAL REVENUE

Total expenditure of the buyer or total revenue of the seller is price × quantity of output bought or sold i.e., TE or TR = p × q. If p falls, q rises if the law of demand holds. But TR or TE may rise, fall or remain the same. That depends on the value of price elasticity of demand.

4. CONSTANT EXPENDITURE OR OUTLAY CURVE

As the very name suggests, constant expenditure or outlay curve is such a curve that expenditure or outlay of the consumer on this curve is constant. In this case, the demand curve is a rectangular hyperbola. Such a curve is also called unit-elastic demand curve.

It is so called because the value of price elasticity of demand at every point on this curve is unity.

5. RELATION AMONG AR, MR AND PRICE ELASTICITY OF DEMAND

There is a standard relation among AR, MR and price elasticity of demand (e_d). It is :

$$MR = AR\left(1-\frac{1}{|e_d|}\right) = p\left(1-\frac{1}{|e_d|}\right)$$ as p is always equal to AR (i.e., $p \equiv AR$).

6. RELATION AMONG TR, MR AND PRICE ELASTICITY OF DEMAND

The relation may be expressed by 3 statements :

(i) When $|e_d| > 1$, MR > 0 and TR will rise with the rise in q or fall in p.

(ii) If $|e_d| < 1$, MR < 0 and TR will fall with the rise in q or fall in p.

(iii) If $|e_d| = 1$, MR = 0, and so TR will remain the same due to rise or fall in price or quantity.

7. ELASTICITY OF FACTOR SUBSTITUTION

The elasticity of factor substitution measures the percentage change in factor proportion due to one percent change in the marginal rate of technical substitution (MRTS).

8. HOMOGENEOUS PRODUCTION FUNCTION AND ITS PROPERTIES

A production function Y = f(K. L) is said to be homogeneous of degree n if $f(\lambda K, \lambda L) = \lambda^n.Y$. The constant n is called the degree of homogeneity. A homogeneous production function has some important properties. **First**, if the production function is homogeneous of degree n, then the marginal productivities of its inputs will be homogeneous of degree (n – 1). **Second**, if the production function is homogeneous of degree 1, then its marginal productivities will be homogeneous of degree zero, or the marginal productivities will depend only on input ratio.

9. HOMOGENEOUS PRODUCTION FUNCTION AND RETURNS TO SCALE

The concept of homogeneous production function may be used to show different types of returns to scale. If the production function is homogeneous of degree 1, it will display constant returns to scale. If the degree of homogeneity is greater than one, it implies increasing returns to scale. If the degree of homogeneity is less than one, it implies decreasing returns to scale.

10. HOMOGENEOUS PRODUCTION FUNCTION AND PRODUCT EXHAUSTION THEOREM

If the production function is homogeneous of degree one and if the factors of production are paid according to their marginal productivities, then total product will be exhausted. This is known as Euler's theorem or product exhaustion theorem.

11. COBB-DOUGLAS PRODUCTION FUNCTION AND ITS PROPERTIES

The specific form of the Cobb-Douglas production function is : $q = AK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$, where q = output, K = capital, L = Labour, A stands for technology (A > 0). This function has the following important properties :

(i) This function is homogeneous of degree one or subject to constant returns to scale.

(ii) AP_K, AP_L, MP_K and MP_L all are diminishing.

(iii) MP_K and MP_L are homogeneous of degree zero or they depend only on input ratio.

(iv) The expansion path under this curve is a straight line passing through the origin.

(v) The elasticity of factor substitution is equal to unity.

(vi) Isoquants under Cobb-Douglas production function will be downward sloping and convex to the origin.

The general form of the Cobb-Douglas production function is : $q = AK^{\alpha}L^{\beta}$, $\alpha + \beta \gtreqless 1$. This function is homogeneous of degree $(\alpha + \beta)$. There will be IRS, CRS or DRS according as $(\alpha + \beta) \gtreqless 1$.

12. CES PRODUCTION FUNCTION AND ITS PROPERTIES

The CES production function can be written as, $q = A \quad K \quad (1 \quad)L^{\frac{1}{\;}}$, $0 < \alpha < 1$, $A > 0$, $0 \neq \rho > -1$.

This function has the following characteristics :

(i) It displays CRS or it is homogeneous of degree one.

(ii) MP_K and MP_L are homogeneous of degree zero.

(iii) Total product will be exhausted if factors are paid according to their marginal productivities.

(iv) Marginal productivities are positive but diminishing.

(v) Isoquants will be downward sloping convex .

(vi) Elasticity of factor substitution is constant.

2.13 Exercises

Short Answer Type Questions (Each of 2.5 marks)

1. Define average and marginal functions.
2. Express price elasticity of demand in terms of average and marginal functions.
3. Show that elasticity of cost = MC/AC.
4. Define income elasticity of demand in terms of average and marginal functions.
5. Show that elasticity of consumption with respect to income is the ratio of MPC and APC.
6. What is the shape of a unit elastic demand curve?
7. Give the log-definition of price elasticity of demand.
8. Define income elasticity in terms of logarithms.
9. State the relation among AR, MR and price elasticity of demand.
10. Define elasticity of factor substitution.
11. What is homogeneous production function?
12. What is linearly homogeneous production function?
13. Give the specific form of Cobb-Douglas production function exhibiting CRS.
14. Give the general form of the Cobb-Douglas production function.
15. What is Euler's theorem?
16. State the product exhaustion theorem.
17. What is the value of elasticity of factor substitution under Cobb-Douglas production function?
18. Give the expression of CES production function.
19. Who are the contributors of CES production function?
20. Why is the CES production function so named?
21. What is the value of elasticity of factor substitution under CES production function?
22. What are the parameters in CES production function?
23. What do the parameters of the CES production function represent?
24. Determine the degree of homogeneity in the following cases :

    (i) $q = K^3 + 3K^2L + 3KL^2 + L^3$

    (ii) $q = \frac{K^2}{L^2} + \frac{L^2}{K^2}$

(iii) $q = \sqrt{\lambda K^2 + \beta KL + L^2}$

(iv) $Y = \dfrac{K}{L^2}$

(v) $q = \sqrt[3]{K^2 L}$

(vi) $q = \dfrac{ax^2 + 2hxy + by^2}{cx + dy}$

(vii) $q = (\beta K^{-e} + \alpha L^{-e})^{-\frac{1}{e}}$

(viii) $q = \alpha\sqrt{K} + \beta\sqrt{L}$

(ix) $z = x_1^3 + 4x_1^2 x_2 + 2x_3^3$

(x) $q = x_1^{0.3} x_2^{0.7}$

(xi) $q = 20K^{1.5} L^{0.5}$

(xii) $z = Ax_1^{\alpha} x_2^{1-\alpha}$

(xiii) $q = AK^{\alpha} L^{\beta}$

(xiv) $Y = x_1^{\alpha} x_2^{1-\alpha} + bx_1 + ax_2$

(xv) $z = \dfrac{ax_1}{bx_2^2}$

25. What is the degree of homogeneity of the demand function, $q = AP^{-\alpha} M^{\beta}$?
26. What is the value of MR if AR = 30 and $|e_d| = 2$?
27. What is the value of $|e_d|$ if AR = 100 and MR = 75?
28. What is the value of AR if MR = 100 and $|e_d| = 5$?
29. What is the value of p if MR = 200 and $|e_d| = 5$?
30. What is the value of $|e_d|$ if p = 30 and MR = 0?

Medium Answer Type Questions

1. Define average and marginal functions. How can they be used to determine elasticity? Give examples.
2. The demand function is : $q = ap^{-\alpha}$. Determine price elasticity of demand.
3. The demand function is : $D = 74 - 2p - p^2$. Calculate price elasticity of demand when D = 50.

4. The demand function is : $q = Ap^{-\alpha}M^{\beta}$ (where q = amount of demand, p = price, M = Income and A, α and β are constants). Determine price and income elasticities of demand.
5. Calculate price elasticity of demand in the following cases :

   (i) $p = \frac{150}{x^2}$ (ii) $x = \frac{60}{p^2}$ (iii) px = 120

6. Calculate price elasticity of demand if the demand function is $x = \frac{200}{p^{\frac{5}{2}}}$.
7. Let the demand function be : aq + bp – k = 0

   If MR = 0, what is the value of e_d?
8. The demand function is : $q = \frac{6y^2}{p} + y$ where y = income, show that $1 < e_y < 2$.
9. Deduce the relation among AR, MR and $|e_d|$.
10. Determine the slope of an indifference curve from a given utility function and define MRS.
11. How can you derive the slope of an isoquant from the production function? Define MRTS.
12. Prove that if a production function is homogeneous of degree n, then the marginal productivities of inputs will be homogeneous of degree (n – 1).
13. Show that if the production function is homogeneous of degree 1, then marginal productivities of its inputs will be homogeneous of degree zero, (or the marginal productivities of its inputs will depend only on input ratio).
14. How is the concept of homogeneous production function related to the concept of returns to scale?
15. Prove that under Cobb-Douglas production function the elasticity of factor substitution is equal to unity.
16. Show that under Cobb-Douglas production function, isoquants will be downward sloping and convex to the origin.
17. Let $q = AK^{\alpha} L^{1-\alpha}$. Show that AP_K, MP_K, AP_L, MP_L will all be diminishing.
18. If the production function is $Y = x_1^{\alpha} x_2^{1-\alpha}$, then prove that total product Y will be just exhausted if factors are paid according to their marginal productivities.

19. Let the production function be $Z = Ax_1^a x_2^b$. What do a and b represent?
20. Show that the degree of homogeneity under CES production function is equal to unity.
21. Prove that under CES production function, the elasticity of factor substitution is a constant.
22. The production function is $x = K^{0.75} L^{0.25}$. Show that product exhaustion theorem holds in this case.
23. Show that under Cobb-Douglas production function, the expansion path of a firm will be a straight line passing through the origin provided input prices are fixed.
24. Let $q = AK^{\alpha}L^{\beta}$. Deduce the expansion path of the firm taking given prices of K and L.
25. Determine the elasticity of factor substitution for the function : $z = cx_1^{\alpha}x_2^{\beta}$ where x_1 and x_2 are the amounts of two factors, X_1 and X_2, respectively.

Long Answer Type Questions (Each of 10 marks)

1. Show that under diminishing MRS, an indifference curve will be strictly convex.
2. Examine the relation between price elasticity of demand and total expenditure of a buyer.
3. Write a short note on constant expenditure or outlay curve.
4. Show that under the assumption of diminishing MRTS an isoquant will be convex to the origin.
5. What is a homogeneous production function? State and prove its two major properties.
6. State and prove the product exhaustion theorem.
7. Prove that under Cobb-Douglas production function, (i) expansion path will be a straight line passing through the origin, and (ii) elasticity of factor substitution will be equal to unity.
8. State and prove two major properties of the CES production function.
9. Show that under CES production function, the elasticity of factor substitution will be a constant.
10. Prove that under CES production function, (i) expansion path will be a straight line passing through the origin, and (ii) isoquants will be downward sloping and convex to the origin.
11. Prove that under CES production function the product exhaustion theorem will hold.
12. Prove that under CES production function, marginal productivities of inputs are positive but diminishing.

2.14 References

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Unit 3 ❐ Maxima and Minima (Extrema) of Functions

Structure

3.1 Objectives

After studying the unit, the reader will be able to know

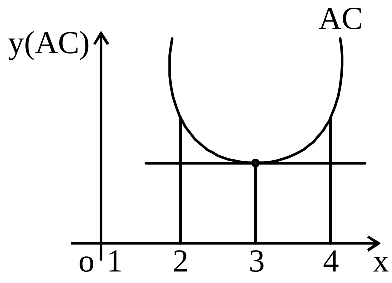
- the maximum and minimum values of a function
- how the maximum and the minimum points can be determined
- the conditions for maximisation and minimisation of a function
- some applications of maxima and minima in Economics

3.2 Introduction

In Economics, we come across so many problems concerned with the target of achieving maximimum or minimum value of a variable. For example, firms, in general, want to maximise their profit. Sometimes a firm wants to maximise sales subject to a minimum profit. Consumers want to maximise their utility subject to a given budget. Similarly, a firm may seek to maximise its output, given the level of cost. In another situation, it may want to produce a given level of output at the minimum possible cost. Planners may want to 'optimise' pollution level, government may want to optimise tax revenue, and so on. Hence we should know how the maximum value or the minimum value of any function can be determined. We should also know the conditions for maximisation and minimisation of an economic variable. This unit seeks to throw light on these issues.

3.3 Concepts of Maxima and Minima of a Single Variable Function

Before considering the concepts of maxima and minima (togetherly called extrema), we should first explain the concepts of increasing and decreasing functions. The function $y = f(x)$ is said to be an increasing function of x if $\frac{dy}{dx}$ or $f'(x) > 0$. On the other hand, the function $y = g(x)$ is called a decreasing function of x if $\frac{dy}{dx} < 0$ or $g'(x) < 0$. For example, in supply function $S = S(P)$, supply(S) is generally an increasing function of price(p). In the simple linear case, we may write, $S = \alpha + \beta p$, (α, β are constants and $\beta > 0$). Hence $\frac{dS}{dp} = \beta > 0$. So, supply(S) is an increasing function of price(p). Taking an example from macroeconomics, we may say that consumption (C) is an increasing function of income (Y). That is, $C = C(Y)$ such that in its linear form, $C = a + bY$, ($a > 0$, $0 < b < 1$). On the other hand, in the demand function, $D = D(P)$, demand (D) is a decreasing function of price (P). In its linear form, we may write, $D = a - bp$, ($a > 0$, b > 0). Here, $\frac{dD}{dp} = -b < 0$. So, demand(D) is a decreasing function of price(P), provided the law of demand holds.



(Fig. 3.1)

Again, a U-shaped curve is first decreasing up to a certain value of the independent variable and then an increasing function beyond that value. For example, a U-shaped AC(y) curve (average cost curve) is first decreasing up to certain level of output (x) and then an increasing function beyond that level of output. In Fig. 3.1, we have given an example.

Example 3.1 : Let $y = 40 - 6x + x^2$ be the equation of an AC curve. Examine whether the function is an increasing or decreasing function at x = 2 and at x = 4.

Solution : We have, $y = 40 - 6x + x^2$

$$\therefore \frac{dy}{dx} = 2x - 6.$$

When x = 2, $\frac{dy}{dx} = 4 - 6 = -2 < 0$

Again, when x = 4, $\frac{dy}{dx} = 8 - 6 = +2 > 0.$

Thus, at x = 2, the AC function is a decreasing function and at x = 4, the AC function is an increasing function.

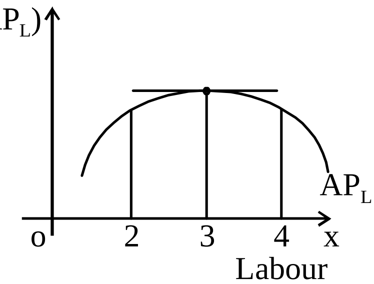
It may be noted that if $\frac{dy}{dx} = 0$, then we have, $2x - 6 = 0$ or, $x = 3$. So, at x = 3, the AC function is neither increasing nor decreasing. At this point the AC function comes to a standstill momentarily. This point is called stationary point and the value of the function at this point is called stationary value. At x = 3, the stationary value of y (= AC) = $40 - 6 \times 3 + 3^2 = 49 - 18 = 31$. We shall take up the issue of stationary point when we shall consider the issue of maxima or minima of a function.

We have mentioned that a u-shaped curve is first decreasing and then increasing after some point. Similarly, an inverted u-shaped curve is first increasing and then decreasing. For example, an inverted AP_L curve (y curve) is first increasing up to a certain level of labour employment (x) and then a decreasing function beyond that level of employment. Let us give an example.

Example 3.2 : Let our average productivity of labour (AP_L) curve be : $y = 40 + 6x - x^2$ where $y = AP_L$ and x = amount of the variable factor, labour. Examine whether the function is an increasing function or a decreasing function at x = 2 and at x = 4.

Solution : We have, $y = 40 + 6x - x^2$

$\therefore \frac{dy}{dx} = 6 - 2x$



(Fig. 3.2)

When x = 2, $\frac{dy}{dx} = 6 - 2 \times 2 = 2 > 0$

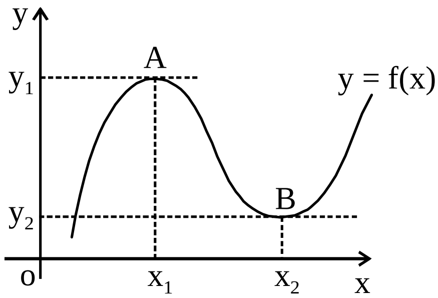
So, at x = 2, the AP_L function is an increasing function.

When x = 4, $\frac{dy}{dx} = 6 - 2 \times 4 = -2 < 0$.

Hence, at x = 4, the AP_L function is a decreasing function.

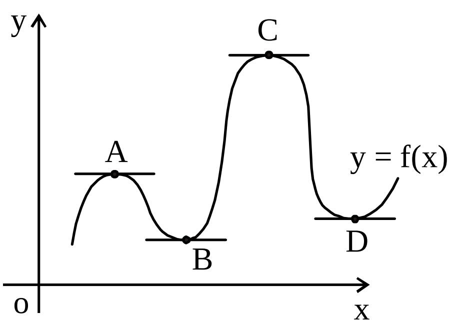
Like our previous function and the related figure, in this case also we see that if $\frac{dy}{dx} = 0$, we have $6 - 2x = 0$ so that x = 3. Thus, at x = 3, the AP_L function is neigher increasing nor decreasing. At x = 3, the point on the AP_L curve is the stationary point. The value of AP_L when x = 3 is : $y = 40 + 6x - x^2 = 40 + 6 \times 3 - 3^2 = 49$. This value of AP_L is called the stationary value of AP_L (= y).

Let us consider the concepts of maxima and minima of a function of one variable. Let y = f(x) is a smooth function i.e., it differentiable everywhere. Its grphical form is given in figure 3.3. From this figure, we see that function y = f(x) has a maximum value at A and a minimum value at B. The maximum and minimum values of a function is called the extreme values y or extrema of the function.



(Fig. 3.3)

We see from the figure that when $x = x_2$, the value of the function f(x) reaches its maximum value, say, y_1. On the other hand, when $x = x_2$, the value of f(x) reaches its minimum. Then the value of y or f(x) is minimum, say, y_2. If the domain (i.e., simply speaking, feasible range of a variable) of the independent variable is quite large, other maximum and minimum values may occur at other points. In our figure 3.4, there are two maximum points, A and C, and two minimum points, B and D. Since point C is the highest maximum point, it is called a global maximum point. The other maximum point A is called local maximum.



(Fig. 3.4)

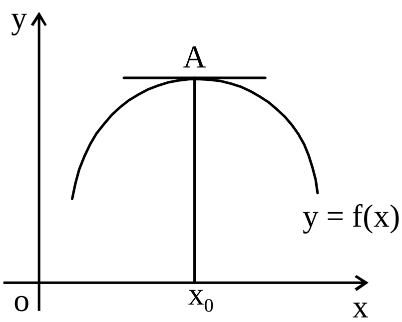
Similarly, point D is called a local minimum point and point B is called a global mimumum point.

It should be noted that at all maximum and minimum points, slope of the function, y = f(x) is zero, i.e, $\frac{dy}{dx}$ or $f'(x) = 0$. Thus, all maximum and minimum points are stationary points. However the converse of this statement is not true i.e., all stationary points will not ncecessarily by maximum or minimum points. That will be clear when we shall consider the concept of point of inflexion in section 3.5.

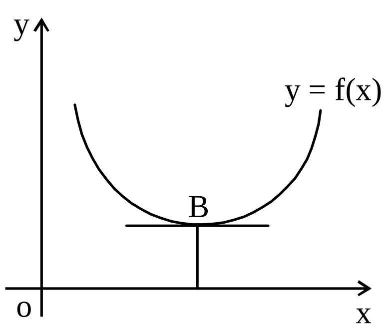
3.4 Identification of Maxima and Minima : First and Second Order Conditions (or Necessary and Sufficient Conditions)

In the previous section we have given some idea about maximum value and minimum value of a function. Let us consider the criteria for the identification of an extremum, i.e., for the maximum or for the minimum. We here mention two alternative criteria for this identification of the type of extremum. The simplest method of identification of the maximum or the minimum value of a function is to observe the pattern of change of the slope of the function.

Consider figure 3.5 and figure 3.6. We see that the function y = f(x) is maximum at A in the first figure while y = f(x) is minimum at B in the second figure. However, in both cases, $\frac{dy}{dx} = f'(x) = 0$. So this is necessary condition for a function to be either maximum or minimum i.e., to be an extremum. This condition is, however, not sufficient, since $\frac{dy}{dx}$ or $f'(x) = 0$ is the condition for maximum or minimum value of the function.



(Fig. 3.5)



(Fig. 3.6)

Hence a sufficient condition is required for the identification of maximum or minimum value. In order to find out this sufficient condition, we have to observe, as we have already mentioned, the pattern of change of the slope of the function. In the figure 3.5, we see that as we move from point O to x_0, the slope of the curve y = f(x) gradually falls and ultimately becomes zero i.e., $\frac{dy}{dx} = f'(x) = 0$ at $x = x_0$. As the slope gradually falls, we can say that change in slope is negative, i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ or $f''(x) < 0$. This is the sufficient condition for maximisation.

Now, consider the sufficient condition for minimisation. In out figure 3.6, the curve y = f(x) is first a decreasing function up to $x = x_0$. So within the range 0 to x_0, $\frac{dy}{dx}$ or $f'(x)$ is negative. Now as we move from point 0 to x_0, the tangents drawn on the curve becomes flatter and flatter. So, their absolute slopes fall. But the slopes are negative and hence we shall say that slopes with negative sign are increasing. Hence, for minimisation, the second condition is that the change in slope of the function y = f(x) should be positive i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ or $f''(x) > 0$. This is the sufficient condition for function y = f(x) to be minimum. Thus, to summarise our results, for a function y = f(x),

Necessary condition : (i) for maximum : $\frac{dy}{dx} = f'(x) = 0$

(ii) for minimisation : $\frac{dy}{dx} f'(x) = 0$

Sufficient condition : (i) for maximum : $\frac{d^2y}{dx^2} = f''(x) < 0$

(ii) for minimum : $\frac{d^2y}{dx^2} = f''(x) > 0$.

The necessary condition, $\frac{dy}{dx} = f'(x) = 0$ is usually called the first order conditon since it is based on the first order derivative of the function y = f(x). The sufficient condition $\frac{d^2y}{dx^2} = f''(x) \gtrless 0$ is called the second order condtion.

To clarify the identification or calculation of maximum or minumum value, we give an example.

Example 3.3 : Find the maximum and minimum values of the expression $x^3 - 3x^2 - 9x + 30$

Solution : Let $y = x^3 - 3x^2 - 9x + 30$

For maximum and minimum value of y, the first order or necessary condition is : $\frac{dy}{dx} = 0$

or, $3x^2 - 6x - 9 = 0$
or, $x^2 - 2x - 3 = 0$
or, $x^2 - 3x + x - 3 = 0$
or, $x(x - 3) + (x - 3) = 0$
or, $(x - 3)(x + 1) = 0$
$\therefore$ x = either – 1 or 3

At these points, $\frac{dy}{dx} = 0$. So these are, so far, stationary values. To identify whether they are extrema (i.e., maximum or minimum value of the function), we have to apply the second order or sufficient condition. That is, we have to consider the sign of $\frac{d^2y}{dx^2}$.

Here, $\frac{d^2y}{dx^2} = 6x - 6$. For maximisation, the condition is : $\frac{d^2y}{dx^2} < 0$, and for minimisation, the condition is : $\frac{d^2y}{dx^2} > 0$

Now, if x = –1, then $\frac{d^2y}{dx^2} = 6x - 6 = -6 - 6 = -12 < 0$.

So, x = –1 gives the maximum value of the given expression.
Hence maximum $y = (-1)^3 - 3(-1)^2 - 9(-1) + 30 = -1 - 3 + 9 + 30 = 39 - 4 = 35$

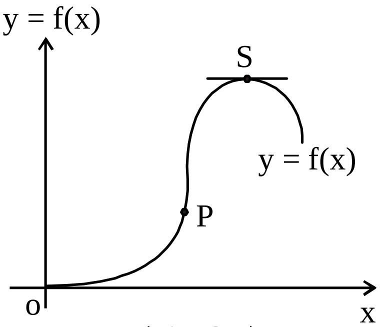
Again, if x = 3, then $\frac{d^2y}{dx^2} = 6 \times (3) - 6 = 18 - 6 = 12 > 0$

So, x = 3 gives a minimum value of the given expression.
The minimum value of $y = (3)^3 - 3(3)^2 - 9(3) + 30 = 27 - 27 - 27 + 30 = 3$

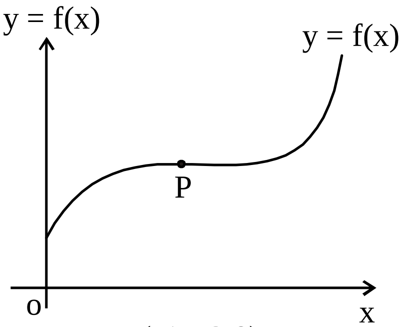
3.5 Point of Inflexion

In terms of a graph or diagram, a point of inflexion on a curve is the point at which the curve changes its curvature. This is a definition in simple terms. We may offer a technical definition of point of inflexion on y = f(x). A point is said to be inflexional if at that point, $f''(x) = 0$ and $f'''(x) \neq 0$. Thus, on the point of inflexion, $f'(x) = 0$ or $f'(x) \neq 0$. It does not impose any restriction on the sign of $f'(x)$. If $f'(x) = 0$, $f''(x) = 0$ and $f'''(x) \neq 0$ at any point on $y = f(x)$, then the point is said to be stationary and inflexional. On the other hand, if $f'(x) \neq 0$, $f''(x) = 0$ and $f'''(x) \neq 0$, at any point on $y = f(x)$, then the value of the function at this point is non-stationary and inflexional. Again, we know that any point on the function $y = f(x)$ at which $\frac{dy}{dx} = 0$, is called a stationary point or a critical point. Thus, a point of inflexion may also be a critical point, but a critical point may not be a point of inflexion. In our figures 3.7 and 3.8, point p is the point of inflexion while in figure 3.7, point S is the critical point or stationary point. There is no critical or stationary point in figure 3.8.

Let us give an example on point of inflexion.



(Fig. 3.7)



(Fig. 3.8)

Example 3.4 : Determine the point of inflexion of the function $y = f(x) = \frac{1}{2}x^4 - 3x^2$

Solution : We have, $y = \frac{1}{2}x^4 - 3x^2$

Here $\frac{dy}{dx} = 2x^3 - 6x$ and $\frac{d^2y}{dx^2} = 6x^2 - 6$.

Now, at the point of inflexion, $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$

We put, $\frac{d^2y}{dx^2} = 0$, So, $6x^2 - 6 = 0$ or, $6(x^2 - 1) = 0$ $\therefore$ $x^2 = 1$ or, $x = \pm 1$.

We see that $\frac{d^3y}{dx^3} = 12x \neq 0$. So, $x = \pm 1$ give the points of inflexion

When $x = +1$, $y = \frac{1}{2}(1)^4 - 3(1)^2 = \frac{1}{2} - 3 = -\frac{5}{2}$

When $x = -1$, $y = \frac{1}{2}(1)^4 - 3(1)^2 = \frac{1}{2} - 3 = -\frac{5}{2}$

So, the points of inflexion are $\left(-1, -\frac{5}{2}\right)$ and $\left(1, -\frac{5}{2}\right)$

3.6 Optimisation of Multivariate Function

The word optimum means the best situation or state of affairs. To achieve an optimum is to optimise, i.e., to maximise or to minimise. So, optimisation is a process or an attempt to achieve or to reach an optimum situation. Economic agents always try to achieve this situation. For example, a consumer tries to maximise (optimise) utility. A firm wants to minimise (optimise) cost or maximise profit. Now, we know that a multivariate function is a function which has more than one independent variable. A special case of a multivariate function means maximisation or minimisation of a function involving two or more independent variables. In this section we shall consider the conditions of maximisation or minimisation of a bivariate function having two independent variables – a special case of multivariate function.

We shall consider the problem of optimisation (i.e., maximisation or minimisation) under two situations : unconstrained optimisation and constrained optimisation. Those techniques have been discussed in the next sections.

3.7 Unconstrained Optimisation or Optimisation without Constraints

In this case, the explanatory variables are independent. Let the bivariate function be $y = f(x_1, x_2)$. This function has the maximum value if (i) $\frac{\partial y}{\partial x_1} = f_1 = 0$ and $\frac{\partial y}{\partial x_2} = f_2 = 0$.

These are necessary or first order conditions. The second order or sufficient conditions are :

(ii) $\frac{\partial^2 y}{\partial x_1^2} \equiv f_{11} < 0$, $\frac{\partial^2 y}{\partial x_1^2} \equiv f_{22} < 0$ and $\frac{\partial^2 y}{\partial x_1^2} \cdot \frac{\partial^2 y}{\partial x_2^2} > \left(\frac{\delta^2 y}{\partial x_1 \partial x_2}\right)^2$

or, $f_{11}.f_{22} > (f_{12})^2$

For minimisation of the function, the necessary conditions or first order conditions are :

(i) $\frac{\partial y}{\partial x_1} = f_1 = 0$ and $\frac{\partial y}{\partial x_2} = f_2 = 0$. The second order or sufficient conditions are :

(ii) $\frac{\partial^2 y}{\partial x_1^2} \equiv f_{11} > 0$, $\frac{\partial^2 y}{\partial x_2^2} = f_{22} > 0$ and $\frac{\partial^2 y}{\partial x_1^2} \cdot \frac{\partial^2 y}{\partial x_2^2} > \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2$ or, $f_{11}.f_{22} > (f_{12})^2$

Let us consider the case when $f_{11}\ f_{22} < (f_{12})^2$.

We know that when $f_1 = 0$ and $f_2 = 0$ it implies a stationary point of a bivariate function. Under this situation, if

(i) $f_{11}f_{22} < f_{12}^2$ and f_{11} and f_{22} have different signs, the function will have a saddle point at that situation.

(ii) $f_{11}f_{22} < f_{12}^2$ and f_{11} and f_{22} have the same sign, the function will have an inflexion point.

If $\frac{\partial^2 y}{\partial x_1^2} \cdot \frac{\partial^2 y}{\partial x_2^2} = \left(\frac{\partial^2 y}{\partial x_1 \partial y_2}\right)^2$ i.e., if $f_{11}f_{22} = f_{12}^2$, the test becomes inconclusive. We cannot say anything definitely.

Consider the following examples.

Example 3.5 : Find the maximum or minimum of the function,

$z = 3x^2 + 2y^2 - xy - 4xy - 7y + 12$

Solution : We have, $z = 3x^2 + 2y^2 - xy - 4x - 7y + 12 = f(x, y)$.

We have, $\frac{\partial z}{\partial x} \equiv f_x = 6x - y - 4$ and $\frac{\delta z}{\partial y} = f_y = 4y - x - 7$

Putting $\frac{\partial z}{\partial x}$ or $f_x = 0$ and $\frac{\partial z}{\partial y} = f_y = 0$, we get, $6x - y = 4$, $4y - x = 7$

Solving them, we get, $x = 1$, $y = 2$.

Thus, (1, 2) is a stationary point or a critical point. It is a point at which there is a possibility of a maximum or a minimum. To know that difinitely, the second order or sufficient condition is to be checked.

Here we have, $\frac{\partial^2 z}{\partial x^2} = f_{xx} = 6 > 0$, $\frac{\partial^2 z}{\partial y^2} = f_{yy} = 4 > 0$, $\frac{\partial^2 y}{\partial x \partial y} = -1$

So we can write, $\left(\frac{\partial^2 y}{\partial x^2}\right) \cdot \left(\frac{\partial^2 z}{\partial y^2}\right) > \left(\frac{\partial^2 z}{\partial x.\partial y}\right)^2$ [Q $6 \times 4 > (-1)^2$]

So, there is a minimum of the function at the point (1, 2). The minimum value of z is,

$z = 3x^2 + 2y^2 - xy - 4x - 7y + 12$

$= 3(1)^2 + 2.(2)^2 - 1 \times 2 - 4 \times 1 - 7 \times 2 + 12$

$= 3 + 8 - 2 - 4 - 14 + 12 = 23 - 20 = 3$

Example 3.6. : Check for the maximum or minimum for the function :

$z = 4x^2 - xy + y^2 - x^3$.

Solution : For maximum or minimum of z, our first order conditions are :

$\frac{\partial z}{\partial x} = 0$, or, $8x - y - 3x^2 = 0$, and $\frac{\partial z}{\partial y} = 0$, or $-x + 2y = 0$

or, $x = 2y$.

Putting this value in the earlier condition, we get,

$8(2y) - y - 3(2y)^2 = 0$

or, $16y - y - 12y^2 = 0$

or, $12y^2 - 15y = 0$ or, $3y(4y - 5) = 0$

$\therefore$ $y = 0$ and $y = \frac{5}{4}$

Now, $x = 2y$ $\therefore$ $x = 0$ or $x = \frac{10}{4} = \frac{5}{2}$ respectively.

Thus, we have, ($x = 0$ and $y = 0$) and $\left(x = \frac{5}{2}, y = \frac{5}{4}\right)$.

Thus, we have two stationary or critical points.

Again, $\frac{\partial^2 z}{\partial x^2} = 8 - 6x$, $\frac{\partial^2 z}{\partial y^2} = 2$ and $\frac{\partial^2 z}{\partial x.\partial y} = -1$.

At the point (0, 0), $\frac{\partial^2 z}{\partial x^2} = 8 > 0$, $\frac{\partial^2 z}{\partial y^2} = 2 > 0$ and

$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x.\partial y}\right)^2$ [as, $8 \times 2 > (-1)^2$]

So, z is minimum at the point (0, 0).

The minimum value of $z = 4x^2 - xy + y^2 - x^3 = 0$.

Let us consider the situation at the point $\left(\frac{5}{2}, \frac{5}{4}\right)$.

We have, $\frac{\partial^2 z}{\partial x^2} = 8 - 6x = 8 - 6 \times \frac{5}{2} = 8 - 15 = -7 < 0$,

$$\frac{\partial^2 z}{\partial y^2} = 2 > 0, \text{ and } \frac{\partial^2 z}{\partial x.\partial y} = -1$$

$$\text{so, } \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) < \left(\frac{\partial^2 z}{\partial x.\partial y}\right)^2 \quad [\text{as } -7 \times 2 < (-1)^2]$$

This implies that there is neither a maximum nor a minimum at the point $\left(\frac{5}{2}, \frac{5}{4}\right)$, i.e., it is a saddle point. At this point, the value of z is :

$$z = 4x^2 - xy + y^2 - x^3 = 4\left(\frac{5}{2}\right)^2 - \frac{5}{2} \times \frac{5}{4} + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{2}\right)^3$$

$$= 25 - \frac{25}{8} + \frac{25}{16} - \frac{125}{8} = \frac{400 + 25 - 50 - 250}{16} = \frac{125}{16} = 7.8125$$

3.8 Constrained Optimisation

Constrained optimisation means the maximisation or minimisation of an objective function where the choice variables are subject to some constraint. In this case, the choice variables are not independent– there is some relation among them.

Examples of constrained optimisation are : utility maximisation subject to a budget constraint, output maximisation subject to a cost constraint, cost minimisation subject to an output constraint, etc.

There are two ways of solving a constrained optimisation problem :

(i) Method of substitution

(ii) Method Lagrange multiplier

The method of substitution can be applied if the objective function which is to be optimised, can be expressed as a function of only one variable by eliminating other variables. Here, elimination is done by using the constraint. If this elimination cannot be done, we have to optimise the objective function by applying Lagrange multiplier method. We shall consider these two methods one by one.

3.8.1 Method of Substitution

The method of substitution is a technique of optimisation under constrained optimisation. This technique is simple to apply and easy to understand. In this method, the objective function is first reduced to a function of single variable by elimination method. After that, the optimisation technique of single variable is applied. The elimination process involves two steps. The first step is to express one of the variables in the constraint explicity in terms of the other variable. The second step is to substitute the value of this variable in the objective function which is to be optimised (i.e., to be maximised or minimised). Then the objective function becomes a function of one variable. It is then simple to optimise this function, as it is done in the case of any single variable function.

We express our argument in mathematical notations. Suppose we want to maximise the function Z = F(x, y) subject to the condition that y = f(x). Then, by substitution we can write, Z = F[x, f(x)] = h(x). Thus, z becomes a function of x only. Now, we have to optimise (maximise or minimise) Z. If Z is to be maximised, we have to apply the first order condition, $\frac{Z}{x} = 0$ and second order condition, $\frac{\partial^2 Z}{\partial x^2} < 0$. If Z is to be minimised, we have to apply the first order or necessary condition, $\frac{\partial Z}{\partial x} = 0$ and the second order condition, $\frac{\partial^2 Z}{\partial x^2} > 0$.

Let us give an example.

Example 3.7. : Optimise $z = x^2 + y^2$ subject to the condition that $2x - y - 5 = 0$

Solution : From the constraint or the condition $2x - y - 5 = 0$, we have, $y = 2x - 5$. This is the first step. Next we substitute this value of y in the objective function, $z = x^2 + y^2 = x^2 + (2x - 5)^2 = x^2 + 4x^2 - 20x + 25$ or, $z = 5x^2 - 20x + 25 = f(x)$

Now, to optimise z, the first order condition or necessary condition is : $\frac{dz}{dx} = 0$,

or, $10x - 20 = 0$ or, $x = 2$. The second derivative, $\frac{\partial^2 z}{\partial x^2} = 10 > 0$. So, in this case, z is minimum at x = 2. Then the minimum value of $z = 5x^2 - 20x + 125$

$= 5(2)^2 - 20 \times 2 + 25 = 20 - 40 + 25 = 5$

Alternatively, $z = x^2 + y^2 = (2)^2 + (-1)^2 = 4 + 1 = 5$

We consider another example.

Example 3.8 : Optimise $z = 60y - 2x^2 + 150$ subject to the constraint : $x - y = 5$

Solution : From the constraint, we get, $x = y + 5$.

We put this value of x in our objective function z.

So, $z = 60y - 2(y + 5)^2 + 150$

$= 60y - 2(y^2 + 10y + 25) + 150 = 60y - 2y^2 - 20y - 50 + 150$

$\therefore z = 40y - 2y^2 + 100$. Thus, z is a function of y only.

Now, we optimise z. The first order condition or necessary condition is : $\frac{dz}{dy} = 0$

or, $40 - 4y = 0$ or, $4y = 40$ $\therefore$ $y = 10$

Here the second derivative of z is : $\frac{d^2z}{dy^2} \quad \frac{d}{dy}\frac{dz}{dy} = -4 < 0$

So, in this case, z is maximum at y = 10

Then $x = y + 5 = 10 + 5 = 15$

Now, we can easily find out the maximum value of z.

Maximum $z = 40y - 2y^2 + 100 = 40 \times 10 - 2(10)^2 + 100 = 400 + 100 - 200 = 300$

Alternatively, we have, $z = 60y - x^2 + 150$.

Putting x = 15 and y = 10, we get,

$z = 60 \times 10 - 2 \times (15)^2 + 150 = 600 - 450 + 150 = 150 + 150 = 300$

3.8.2 Lagrange Multiplier Method

In the case of constrained maximisation or minimisation of a function, we cannot apply the simple technique of necessary and sufficient conditions. Here we have to maximise or minimise a function (called objective function) subject to certain restriction(s) called constraint(s). Hence the problem is called the problem of constrained maximisation or the problem of constrained minimisation. In this case, an alternative technique is used. Either we shall incorporate the constraint into the objective function, or we shall follow a different technique called Lagrange technique. The former is called the substitution method which we have discussed in the previous section. We shall now consider the second method which is formally called Lagrange multiplier method.

The Langrange multiplier method is an optimisation technique where we have to optimise (maximise or minimise) an objective function subject to a given constraint. Here the variables of the constraint are so related that one cannot be explicitly expressed in terms of other(s). Let us discuss the lagrange multiplier method in details.

Suppose we have to optimise a bivariate function $y = f(x_1, x_2)$. This is our objective function. We have to maintain a restriction which is called constraint. We assume that the constraint involves two variables and it is given in an implict from : $h(x_1, x_2) = k$ where k is a constant. So, our problem stands as :

Optimise $y = f(x_1, x_2)$ subject to $h(x_1, x_2) = k$. To solve this problem by Lagrange method, we first construct an auxiliary objective function. This auxiliary objective function is obtained by adding the original objective function with a Lagrange multiplier(λ) multiplied with the constraint in the form of zero. Thus, the auxiliary Lagrange function, say, V is given by the following expression,

$V = f(x_1, x_2) + \lambda[k - h(x_1, x_2)]$

Here the objective function V becomes a function of three variables, namely, x_1, x_2 and λ. Now the problem of constrained optimisation has become a problem of unconstrained optimisation. So, to maximise V, the first order or necessary conditions require,

$$\frac{\partial V}{\partial x_1} = 0, \quad \text{or, } f_1 - \lambda h_1 = 0$$

$$\frac{\partial V}{\partial x_2} = 0, \quad \text{or, } f_2 - \lambda h_2 = 0$$

$$\frac{\partial V}{\partial \lambda} = 0, \quad \text{or, } k - h(x_1, x_2) = 0$$

We should note that the third equation is actually the given constraint. It implies that in our optimisation process, we are obeying the restriction put by the constraint. Now, from the first two equations, we get, $\frac{f_1}{h_1} = \lambda$ and $\frac{f_2}{h_2} = \lambda$

So, $\frac{f_1}{h_1} = \frac{f_2}{h_2} = \lambda$.

The new objective function V is optimised at the point where this condition is satisfied. Again, if the constraint is satisfied i.e., $h(x_1, x_2) = k$ or, $k - h(x_1, x_2) = 0$, then optimisation of V implies optimisation of y. Thus, where the condition $\frac{f_1}{h_1} = \frac{f_2}{h_2} = \lambda$ is satisfied, the objective function $y = f(x_1, x_2)$ is automatically optimised.

We give two illustrations below. We take same earlier two problems solved by the substitution method. Now we shall solve those optimisation problems by following Lagrange multiplier method. We shall see that both methods give the same result.

Examples 3.9 : Following Lagrange multiplier method, optimise $z = x^2 + y^2$ subject to the condition that $2x - y - 5 = 0$

[This is our earlier problem No. 3.7.]

Solution : To solve this problem by Lagrange method we construct the Lagrange expression, say, V. Our problem is to optimise $z = x^2 + y^2$ subject to $2x - y - 5 = 0$. So, the Lagrangian function (V) is :

$V = x^2 + y^2 + \lambda(2x - y - 5)$ where λ is the Lagrange multiplier. Here $V = V(x, y, \lambda)$. So, to optimise V, first order conditions require,

$\frac{\partial V}{\partial x}$ or $V_x = 0$ or, $2x + 2\lambda = 0$ ...(a)

$\frac{\partial V}{\partial y}$ or $V_y = 0$ or, $2y - \lambda = 0$ ...(b)

$\frac{\partial V}{\partial \lambda}$ or $V_\lambda = 0$ or, $2x - 2y - 5 = 0$ ...(c)

We have three equations and three unknows, namely, x, y and λ.
So, we can solve for them. From (a) we get, $\lambda = -x$ and from (b) we get, $\lambda = 2y$.
Comparing them, we get, $-x = 2y$ or, $x = -2y$
Putting this value of x in (c) we get, $2(-2y) - y - 5 = 0$ or, $-5y = 5$ or, $y = -1$
$\therefore$ $x = -2(-1) = 2$ and $\lambda = 2y = 2(-1) = -2$
So the optimum value of $z = x^2 + y^2 = (2)^2 + (-1)^2 = 4 + 1 = 5$.

In example 3.7, we obtained the same optimum value of z. We, however, here assume that second order conditions are fulfilled.

We now consider the example given in 3.8.

Example 3.10 : Applying Lagrange technique optimise $z = 60y - 2x^2 + 150$ subject to the constraint : $x - y = 5$

Solution : Here our problem is to optimise $z = 60y - 2x^2 + 150$ subject to $x - y = 5$ or $x - y - 5 = 0$. So it is a problem of constrained optimisation. We form the Lagrange expression.

$v = 60y - 2x^2 + 150 + \lambda(x - y - 5)$

First order conditions or necessary conditions require,

$\frac{\partial v}{\partial x} = 0$ or, $-4x + \lambda = 0$ ... (a)

$\frac{\partial v}{\partial y} = 0$ or, $60 - \lambda = 0$ ... (b)

$\frac{\partial v}{\partial \lambda} = 0$ or, $x - y - 5$ ... (c)

Solving these 3 equations, we shall get the values of 3 variables, namely, x, y and λ.
From (a) we get, $\lambda = 4x$ and from (b) we get $\lambda = 60$
$\therefore$ $4x = 60$ or, $x = 15$.
Putting this value in (c), we get, $15 - y - 5 = 0$ or, $y = 10$
So, the optimum value of $z = 60y - 2x^2 + 150$

$= 60 \times 10 - 2(15)^2 + 150 = 600 + 150 - 450 = 300$

We got the same optimum value in example 3.8. We assume that the second order conditions of optimisation have been fulfilled.

It may be noted that in the last two examples adopting Lagrange technique we have obtained the extrema of the given function. We could not say whether these values are maxima or minima. This is because, in the above two examples we have applied first order or necessary conditions for optimisation. They are not sufficient for maximisation or minimisation. The sufficient condition can be obtained from the second order conditions.

3.9 Sufficient Condition for Constrained Optimisation

Consider a bivariate function : $y = f(x_1, x_2)$. If there is no constraint or restriction i.e., if the optimisation problem is unconstrained, we have a simple problem of optimisation. In that case, our second order or sufficient condition for optimisation is = $d^2y < 0$ for a maximum and $d^2y > 0$ for a minimum.

Consider now the case of a constrained optimisation. Suppose we want to optimise a function $y = f(x_1, x_2)$ subject to the restriction or constraint : $h(x_1, x_2) = k$ where k is a constant. In this case of constrained optimisation, the case is not so simple. In the case of an unconstrained optimisation, the constraint is absent. Hence we can consider changes is x_1 and x_2 (i.e., dx_1 and dx_2) as arbitrary changes. But in the case of constrained optimisation, both dx_1 and dx_2 can be taken as arbitrary changes. Here, either we have to assume that x_1 depends on x_2, or the other way round i.e., x_2 depends on x_1. Thus, if we consider dx_1 as an arbitrary change, dx_2 must be assumed to be dependent on dx_1. Similarly, if we take dx_2 as an arbitrary change, dx_1 must be assumed to be dependent on dx_2.

Under constrained optimisation, our constraint is given as : $h(x_1, x_2) = k$ where k is a constant. Then by total derivative and putting $dk = 0$, we get, $h_1dx_1 + h_2dx_2 = 0$. In this case, the sufficiency condition for having a maximum or minimum will be changed. An extremum will be a point of maximum if $d^2y < 0$ subject to the restriction $dh = 0$ and it will be a point of minimum if $d^2y > 0$ subject to the restriction that $dg = 0$. The ultimate expressions of these conditions for maximisation and minimisation can be conveniently represented in terms of determinant, or more specifically, in terms of a Hessian Bordered determinant. We have considered these concepts in unit 5.

3.10 Applications of Maxima and Minima in Economics

The concepts of Maxima and Minima have so many applications in Economics. We mention below some major such applications.

3.10.1 Rleation between Average Product (AP) and Marginal Product (MP) of an Input

Let the total product function be q = f(L) where q = total output and L = labour. So, average product of labour = $AP_L = \frac{q}{L} = \frac{f(L)}{L}$. Marginal product of labour (MP_L) is, in terms of calculus, the first derivative of the total product function i.e., $MP_L = \frac{dq}{dL} = f'(L)$.

There is a standard relation between AP_L and MP_L. Let us try to derive that relation between.

We know, $AP_L = \frac{q}{L} = \frac{f(L)}{L} = \phi(L)$.

Now, slope of AP_L curve = $\frac{dAP_L}{dL} = \frac{\frac{dq}{dL}.L - q}{L^2}$

Thus, slope of AP_L curve $\gtreqless 0$ according as $\frac{dq}{dL}L \gtreqless q$

or, according as $\frac{dq}{dL} \gtreqless \frac{q}{L}$

or, according as $MP_L \gtreqless AP_L$.

Thus, slope of AP_L curve > 0 i.e., AP_L rises when $MP_L > AP_L$.

Again, slope of AP_L curve = 0 i.e., AP_L is constant or maximum when $MP_L = AP_L$. Similarly, slope of AP_L curve < 0 i.e. AP_L falls when $MP_L < AP_L$.

This is the standard relation between AP and MP.

The relation can also be established in an alternative manner. We may write, TP = $AP_L \times L$, i.e., q = $AP_L \times L$. Now, both q and AP_L are functions of L, so, differentiating both sides with respect to L, we get,

$\frac{dq}{dL} = AP_L \times 1 + L.\frac{dAP_L}{dL}$ or, $MP_L = AP_L + L \times$ (slope of AP_L curve)

Now, when AP_L rises, slope of AP_L curve or $\frac{dAP_L}{dL} > 0$. So, $MP_L > AP_L$.

When AP_L falls, slope of AP_L curve or $\frac{dAP_L}{dL} < 0$. So, $MP_L < AP_L$. When AP_L is maximum or constant slope of AP_L curve or $\frac{dAP_L}{dL} = 0$. Then $MP_L = AP_L$.

3.10.2 Relation between Average Cost (AC) and Marginal Cost (MC) of output

Let the total cost function of the firm be C = f(q) where C = total cost and q = output. Now, average cost. (AC) is the cost per untit of output, i.e., AC = $\frac{C}{q} = \frac{f(q)}{q}$ = g(q). On the other hand, marginal cost is the first order derivative of the total cost function, i.e., MC = $\frac{dC}{dq}$ = C′(q).

Now there is a standard relation between AC and MC. Let us try to derive this relation.

We may write, total cost, C = q × AC.

Both C and AC are functions of q(output). So, we can differentiate both sides with respect to q. Then we get, $\frac{dC}{dq} = AC \times 1 + q.\frac{dAC}{dq}$ i.e., MC = AC + q × (slope of AC curve).

Now, when AC rises, $\frac{dAC}{dq} > 0$, or slope of AC curve > 0. Then MC > AC.

When AC falls, $\frac{dAC}{dq} \quad 0$, or, slope of AC curve < 0.

Then MC < AC.

When AC is minimum or remains constant, $\frac{dAC}{dq} = 0$ or, slope of AC curve = 0.

Then MC = AC.

We may prove this relation between AC and MC in an alternative manner. We may write, AC = $\frac{\text{total cost}}{\text{output}}$ i.e., AC = $\frac{C}{q} = \frac{C(q)}{q}$.

Now, differentiating both sides with respect to q, we get,

$$\frac{dAC}{dq} = \frac{\frac{dC}{dq}.q - C \times 1}{q^2} = \frac{\frac{dC}{dq}.q - C}{q^2}$$

So, $\frac{dAC}{dq}$ or slope of AC curve $\gtreqless 0$ according as $\frac{dC}{dq}.q \gtreqless C$

or, according as, $\frac{dC}{dq} \gtreqless \frac{C}{q}$ or, according as, MC $\gtreqless$ AC.

Thus, $\frac{dAC}{dq}$ or slope of AC curve > 0 i.e., AC rises when MC $\gtreqless$ AC.

Similarly, $\frac{dAC}{dq}$ or slope of AC curve < 0 i.e., AC falls when MC < AC.

Again, $\frac{dAC}{dq}$ or slope of AC curve = 0, i.e., AC is stationary or AC is minimum or constant when MC = AC.

3.10.3 Profit Maximisation by a Firm

We first consider the conditions for profit-maximising employment of a firm. We assume that total output(q) is a function of labour-employment(L) only, i.e., q = f(L). Let the money wage rate per unit of labour be ω and price per unit or output be p. Now, total profit, Π = total revenue(R)– total cost (C). or, Π = R – C. Here, total revenue, R = p.q and total cost, C = TVC + TFC = ω.L + F where F = TFC.

So, Π = R – C = p.q – ωL – F = p.f(L) – ωL – F.

Here, p, ω and F are constants. So the whole expression on the RHS is a function of L only. Thus, we get, Π = Π(L) i.e., total profit is a function of labour only. Now to maximise Π, the first order condition or the necessary condition is :

(i) $\frac{d\Pi}{dL} = 0$. The second order condition or the sufficient condition is : $\frac{d^2\Pi}{dL^2} < 0$

Now, $\frac{d\Pi}{dL} = p.\frac{dq}{dL} - \omega.1 - 0 = p.\frac{dq}{dL} - \omega$

Putting $\frac{d\Pi}{dL} = 0$, we have, $p.\frac{dq}{dL} - \omega = 0$

or, $p.\frac{dq}{dL} = \omega$ or, $p \times MP_L = \omega$

or, value of the marginal product (VMP) = money wage rate (ω).

It can be rewritten as, $\frac{dq}{dL} = \frac{\omega}{p}$ i.e., marginal physical product of labour should be equal to real wage rate.

The second order condition or the sufficient condition for profit maximisation requires, $\frac{d^2\Pi}{dL^2} < 0$.

Here, $\frac{d^2\Pi}{dL^2} = p.\frac{d^2q}{dL^2} < 0$.

As p > 0, the second order condition for profit-maximising employment requires, $\frac{d^2q}{dL^2} < 0$ or, $\frac{d}{dL}\left(\frac{dq}{dL}\right) < 0$ or, $\frac{d(MP_L)}{dL} < 0$ or, slope of MP_L curve < 0 i.e., MP_L should be diminishing.

Thus, for profit maximisation, employment should be made at the point where the following two conditions are fulfilled :

(i) **First-order condition or necessary condition :** MP_L = real wage rate or, value of marginal product of labour = money wage rate.

(ii) **Second-order condition or sufficient condition :** MP_L should be diminishing or the MP_L curve should be downward sloping.

Example 3.11 : The short run production function is : $q = -0.1L^3 + 6L^2 + 12L$. If wage rate is ` 360 and $p_q$ = ` 30, how much labour(L) will be employed and how much output (q) will be produced in order to maximise profit?

Solution : Profit will be maximum when (i) $MP_L = \frac{\omega}{\rho}$ and (ii) slope of MP_L curve < 0.

Now, we have, $q = -0.11^3 + 6L^2 + 12L$

$\therefore\ MP_L = \frac{dq}{dL} = -0.3L^2 + 12L + 12$. Further, $\frac{\omega}{\rho} = \frac{360}{30} = 12$.

So, putting $MP_L = \frac{\omega}{\rho}$, we get, $-0.3L^2 + 12L + 12 = 12$

or, $-0.3L^2 + 12L = 0$ or, $0.3L^2 + 12L = 0$ or, $L(0.3L - 12) = 0$

$\therefore$ Either L = 0, or, 0.3L – 12 = 0 or, $L = \frac{12}{0.3} = 12 \times \frac{10}{3} = 40$

Thus, from the first order or necessary condition, we get L = 0, 40.

Let us consider the second order or the sufficient condition. It requires that the slope of the MP_L curve should be negative.

Now, slope of MP_L curve = $\frac{d}{dL}\left(\frac{dq}{dL}\right) = \frac{d^2q}{dL^2} = -0.6L + 12.$

If L = 0, slope of MP_L curve = $\frac{d^2q}{dL^2} = -0.6 \times 0 + 12 = 12 > 0.$

If L = 40, slope of MP_L curve = $\frac{d^2q}{dL^2} = -0.6 \times 40 + 12 = -24 + 12 = -12 < 0.$

So, profit will be maximimum if L = 40

Then the amount of profit-maximising output is :

$q = -0.1L^3 + 6L^2 + 12L = -0.1 \times (40)^3 + 6 \times (40)^2 + 12 \times (40)$

$= -6400 + 9600 + 480 = 3200 + 480 = 3680$

Let us suppose that instead of one input, the firm has two variable inputs, namely, capital (K) and labour (L). The output(q) is being sold in a perfectly competitive market so that price of output (p) is fixed. We also assume that price of K(p_K) and price of labour (p_L) are also fixed. So total profit of the firm is given by the expression,

$\Pi = R - C = p \times q - p_K K - p_L L$. Here q = f(K, L)

So, $\Pi = pf(K, L) - p_K K - p_L L$. Thus, Π depends on K and L i.e., Π = Π(K, L). This is a bivariate function without any constraint. We know the conditions of maximisation of this function.

First order or necessary conditions to maximise Π are :

$$\frac{\partial \Pi}{\partial K} = \Pi_K = 0, \text{ or, } p.f_K - p_K = 0 \text{ or, } p.f_K = p_K \qquad ...(a)$$

$$\frac{\partial \Pi}{\partial L} = \Pi_L = 0, \text{ or, } p.f_L - p_L = 0 \text{ or, } p.f_L = p_L \qquad ...(b)$$

Condition (a) states that the value of the margianl product of capital should be equal to the price of capital. Similarly, condition(b) states that the value of the marginal product of labour should be equal to the price of labour.

Second order conditions to maximise Π require, $\frac{\partial^2 \Pi}{\partial K^2} = f_{KK} < 0$, $\frac{\partial^2 \Pi}{\partial L^2} = f_{LL} < 0$ and

$\frac{\partial^2 \Pi}{\partial K^2} . \frac{\partial^2 \Pi}{\partial L^2} > \left(\frac{\partial^2 \Pi}{\partial L \partial K}\right)^2$ or, in alternative symbol, $f_{KK}.f_{LL} > (f_{LK})^2$.

We consider an example.

Example 3.12 : Let the production function of the firm is q = $12-\frac{1}{K}-\frac{1}{L}$ and $p_K = 4$, $p_L = 1$ and $p_q = 9$.

Determine the profit-maximising input conbination and also the amount of profit.

Solution : Total profit, $\Pi = R - C = p_q.q - (p_K K + p_L L)$

$$\therefore \Pi = 9\left(12-\frac{1}{K}-\frac{1}{L}\right)-4K-L$$

$$\Pi = 108-\frac{9}{K}-\frac{9}{L}-4K-L$$

Here Π depends on K and L i.e., $\Pi = \Pi(K, L)$.

The first order conditions to maximise Π require,

$$\Pi_K = \frac{\partial \Pi}{\partial K} = 0, \text{ or, } \frac{9}{K^2}-4=0 \qquad \text{...(a)}$$

$$\Pi_L = \frac{\partial \Pi}{\partial L} = 0, \text{ or, } \frac{9}{L^2}-1=0 \qquad \text{...(b)}$$

From (a) we get, $\frac{9}{K^2}=4$, or, $K^2 = \frac{9}{4}$ $\therefore K = \frac{3}{2}$

From (b) we get, $\frac{9}{L^2}=1$, or, $L^2 = 9$ $\therefore L = 3$

The second order conditions to maximise Π require,

$$\Pi_{KK} = \frac{\partial^2 \Pi}{\partial K^2} < 0,\ \Pi_{LL} = \frac{\partial^2 \Pi}{\partial L^2} < 0 \text{ and } \Pi_{KK}.\Pi_{LL} > (\Pi_{KL})^2.$$

Here, $\Pi_{KK} = -\frac{18}{K^3} = -18\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} = -\frac{16}{3} < 0$

$\Pi_{LL} = -\frac{18}{L^3} = \frac{-18}{3\times3\times3} = -\frac{2}{3} < 0$

Further, Π_{KL} or $\Pi_{LK} = 0$

Now, $\Pi_{KK}.\Pi_{LL} = -\frac{16}{3}\times-\frac{2}{3} = \frac{32}{9} > 0$

i.e., $\Pi_{KK}.\Pi_{LL} > (\Pi_{KL})^2$ or, $\Pi_{KK}\Pi_{LL} > (\Pi_{KL})^2$ (as $\Pi_{KL} = \Pi_{LK}$ by Young's theorem).

So, second order conditions are fulfilled.

$\therefore$ K = $\frac{3}{2}$, L = 3.

$\Pi = 108 - 9\times\frac{2}{3} - \frac{9}{3} - 4\times\frac{3}{2} - 3$ = 108 – 6 – 3 – 6 – 3 = 108 – 18 = 90 (**Ans.**)

Let us consider the conditions for profit-maximising sales of a firm. Profit(Π) = total revenue (R) - total cost(C) i.e., Π = R – C. Now, R = p × q where p = price of output, q = quantity of output. We assume that p is fixed, i.e., firm is selling its output in a perfectly competitive market. Further, total cost is a function of output(q) i.e., C = C(q). So, we have, Π = R – C = p.q – C(q) = Π(q) i.e., Π is a function of output (q).

To maximise Π, the first order condition or the necessary condition is : $\frac{d\Pi}{dq} = 0$,

i.e., p.1 – $\frac{dC}{dq} = 0$ or, p = $\frac{dC}{dq}$.

$\frac{dC}{dq}$ is the marginal cost (MC). So, the first order condition for profit maximisation is : p = MC.

Again, when p is fixed, we have, from R = p.q, $\frac{dR}{dq} = p$ i.e., MR = p. So, the first order condition for profit maximisation under perfect competition can also be written as, MR = MC.

The second order condition or the sufficient condition to maximise Π requires,

$\frac{d^2\Pi}{dq^2} < 0$ i.e., $0 - \frac{d^2C}{dq^2} < 0$ or, $\frac{d^2C}{dq^2} > 0$

Now, $\frac{d^2C}{dq^2} = \frac{d}{dq}\left(\frac{dC}{dq}\right)$ = $\frac{d(MC)}{dq}$ = slope of MC curve.

So, the second order condition for profit maximisation under perfect competition requires that slope of MC curve should be positive i.e., MC curve should be upward rising.

Example 3.12 : A perfectly competitive firm is selling its product at price of ` 5 per unit. Its total cost curve is : C = $q^3 - 10q^2 + 17q + 60$ where 60 = TFC or total fixed cost. Determine the equilibrium output and the amount of maximum profit.

Solution : We have $C = q^3 - 10q^2 + 17q + 60$.

$\therefore$ $MC = \frac{dC}{dq} = 3q^2 - 20q + 17$. Futher, $p = 5$.

Now, the first order condtion for profit maximisation under perfect competition requires, $p = MC$.

So, we can write, $3q^2 - 20q + 17 = 5$

or, $3q^2 - 20q + 12 = 0$

or, $3q^2 - 18q - 2q + 12 = 0$

or, $3q(q - 6) - 2(q - 6) = 0$

or, $(q - 6)(3q - 2) = 0$

So, either $(q - 6) = 0$ or $(3q - 2) = 0$. So, $q = 6$ or $\frac{2}{3}$.

The second order condition under perfect competition requires that

slope of MC curve > 0 or, $\frac{dMC}{dq} > 0$. Here, $\frac{dMC}{dq} = 6q - 20$

If $q = \frac{2}{3}$, we get, $\frac{dMC}{dq} = 6 \times \frac{2}{3} - 20 = 4 - 20 = -16 < 0$

If $q = 6$, we get, $\frac{dMC}{dq} = 6 \times 6 - 20 = 36 - 20 = +16 > 0$

So, profit is maximum if $q = 6$.

Amount of maximum $\Pi = R - C = p \times q - C$

If $q = 6$, total revenue, $R = p \times q = 5 \times 6 = 30$

Total cost $= C = q^3 - 10q^2 + 17q + 60$

$= (6)^3 - 10(6)^2 + 17 \times 6 + 60$

$= 216 - 360 + 102 + 60$

$= 378 - 360 = 18$

So, the amount of profit $= R - C = 30 - 18 = 12$

Alternative method

We may solve the problem in an alternative manner.

We have, $p = 5$. So, $R = pq = 5q$. Further, $C = q^3 - 10q^2 + 17q + 60$.

Now, total profit, $\Pi = R - C = 5q - q^3 + 10q^2 - 17q - 60 = \Pi(q)$ i.e., total profit is a function of output (q).

To maximise Π the first order condition or necessary condition is : $\frac{d\Pi}{dq} = 0$.

Here, $\frac{d\Pi}{dq} = -3q^2 + 20q - 12$

Putting $\frac{d\Pi}{dq} = 0$, $-3q^2 + 20q - 12 = 0$

or, $3q^2 - 20q + 12 = 0$

or, $3q^2 - 18q - 2q + 12 = 0$

or, $3q(q - 6) - 2(q - 6) = 0$

or, $(q - 6)(3q - 2) = 0$

So, either $(q - 6) = 0$, or, $3q - 2 = 0$.

Then, either $q = 6$ or, $q = \frac{2}{3}$

The second order condition or sufficient condition requires,

$\frac{d^2\Pi}{dq^2} < 0$. Here, $\frac{d^2\Pi}{dq^2} = -6q + 20$

If $q = \frac{2}{3}$, $\frac{d^2\Pi}{dq^2} = -6q + 20 = -6\times\frac{2}{3}+20 = 20 - 4 = 16 > 0$

If $q = 6$, $\frac{d^2\Pi}{dq^2} = -6q + 20 = -6 \times 6 + 20 = -36 + 16 = -16 < 0$

So, Π is maximum if $q = 6$

The amount of maximum profit =

$\Pi = -q^3 + 10q^2 - 12q - 60$

$= (-6)^2 + 10(6)^2 - 12 \times 6 - 60 = -216 + 360 - 72 - 60$

$= 360 - 216 - 72 - 60 = 360 - 348 = 12$

Let us consider the conditions for profit maximising sales of a firm when price of output is not fixed. That is, the firm is selling its output in an imperfectly competitive market or the firm is a monopolist. Here, $\Pi = R - C = p \times q - C$. Here $p = f(q)$ is the inverse demand function faced by the monopolist or by any imperfectly competitive firm. So, $R = p.q = f(q).q = R(q)$. We know that total cost(C) depends the level of output i.e., $C = C(q)$. So, $\Pi = R(q) - C(q) = \Pi(q)$ i.e., Π is a function of output(q).

To maximise Π, the first order or the necessary condition requires,

$\frac{d\Pi}{dq} = 0$, or $\frac{dR}{dq} - \frac{dC}{dq} = 0$, or $\frac{dR}{dq} = \frac{dC}{dq}$

Now, $\frac{dR}{dq}$ is MR while $\frac{dC}{dq}$ = MC. So, the first order condition for profit maximisation requires, MR = MC.

We can deduce this condition in a slightly different manner.

We have, Π = R – C = p(q).q – C(q).

Now, putting $\frac{d\Pi}{dq}$ = 0, we get, $\frac{dp}{dq}.q + p - \frac{dC}{dq} = 0$

$$\text{or, } p\left(1+\frac{dp}{dq}.\frac{q}{p}\right)=\frac{dC}{dq} \text{ or, } p\left(1-\frac{1}{-\frac{p}{q}.\frac{dq}{dp}}\right)=\frac{dC}{dq}, \text{ or, } p\left(1-\frac{1}{|e_d|}\right)=\frac{dC}{dq}$$

Now, we know that MR $= p\left(1-\frac{1}{|e_d|}\right)$. So our first order condition becomes, MR = MC.

The second order condition to maximise Π requires, $\frac{d^2\Pi}{dq^2} < 0$, i.e., $\frac{d^2R}{dq^2} - \frac{d^2C}{dq^2} < 0$,

or, $\frac{d^2C}{dq^2} > \frac{d^2R}{dq^2}$ i.e. slope of MC curve > slope of MR curve.

In other words, the MC curve should cut the MR curve from below.

Example 3.14 : The demand function faced by a monopolist or by any imperfectly competitive firm is : p = 80 – 0.2q and the cost function is : C = 50 + 0.05 q^2. Find profit-maximising output, price and profit.

Solution : We have, p = 80 – 0.2q ∴ R = pq = 80q – 0.2q^2

So, MR = $\frac{dR}{dq}$ = 80 – 0.4q

Again, C = 50 + 0.05q^2 ∴ MC = $\frac{dC}{dq}$ = 0.1q.

Now the first order condition or necessary condition requires, MR = MC

or, 80 – 0.4q = 0.1q or, 0.5q = 80 ∴ q = $\frac{80}{0.5} = 160$

Here, slope of MC curve = $\frac{dMC}{dq}$ = 0.1 and slope of MR curve = $\frac{dMR}{dq}$ = – 0.4.

The second order condition for profit maximisation requires,

slope of MC curve > slope of MR curve.

As 0.1 > – 0.4, the second order condition is fulfilled.

$\therefore$ q = 160. Then p = 80 – 0.2q = 80 – 0.2 × 160 = 80 – 32 = 48

Total revenue, R = p × q = 48 × 160 = 7680.

Total cost, C = 50 + 0.05 q^2 = 50 + 0.05 × 160 × 160

= 50 + 8 × 160 = 50 + 1280 = 1330.

So Π = R – C = 7680 – 1330 = 6350. This is the amount of maximum profit.

Alternative method :

We may solve this problem also by following our alternative method.

We have, R = p.q = (80 – 0.2q)q = 80q – $0.2q^2$ and C = 50 + $0.05q^2$.

Now, total profit, Π = R – C = 80q – $0.2q^2$ – 50 – $0.05q^2$

or, Π = 80q – $0.25q^2$ – 50.

Thus, Π depends on or is a function of output(q). So, the first order condition or the necessary condition is, $\frac{d\Pi}{dq} = 0$

Here, $\frac{d\Pi}{dq}$ = 80 – 0.5q. Putting $\frac{d\Pi}{dq}$ = 0, we get, 80 – 0.5q = 0

or, 0.5q = 80 $\quad\therefore$ q = $\frac{80}{0.5}$ = 160

The second order condition or the sufficient condition requires, $\frac{d^2\Pi}{dq^2} < 0$.

Here, $\frac{d^2\Pi}{dq^2}$ = – 0.5 < 0. So, the second order condition is fulfilled. Hence, profit is maximum if q = 160. Then price, p = 80 – 0.2q = 80 – 0.2 × 160 = 80 – 32 = 48

Then total profit, Π = 80q – $0.25q^2$ – 50

= 80 × 160 – 0.25 × 160 × 160 – 50

= 12800 – 6400 – 50

= 12800 – 6450 = 6350

3.10.4 Utility Maximisation with Budget Constraint

Suppose we have a bivariate utility function U = f(q_1, q_2). We have to maximise utility(U) subject to the constraint, M = $p_1q_1 + p_2q_2$. So, it is a case of constrained maximisation. We can do this by two alternative methods. One is the substitution method and the other is the Lagrangean method. We shall first consider the method of substitution.

Substitution method

In this method, we express one of the variables in the constraint explicitly in terms of the other variable. Then we substitute the value of this variable in the objective function (i.e., our utility function) which is to be maximised in this case.

Now, one constraint is , $p_1q_1 + p_2q_2 = M$. It can be rewritten as, $p_2q_2 = M - p_1q_1$

or, $q_2 = \dfrac{M - p_1q_1}{p_2}$. We incorporate this value of q_2 into the utility function. Then we get, $U = f(q_1, q_2) = f\left(q_1, \dfrac{M - p_1q_1}{p_2}\right)$. Thus, U becomes a function of q_1 alone.

The first order condition or the necessary condition to maximise U requires, $\dfrac{dU}{dq_1} = 0$

Now, $\dfrac{dU}{dq_1} = f_1 + f_2\left(-\dfrac{p_1}{p_2}\right)$ ... (1)

So, $f_1 + f_2\left(-\dfrac{p_1}{p_2}\right) = 0$ or, $f_1 = f_2.\dfrac{p_1}{p_2}$ or, $\dfrac{f_1}{f_2} = \dfrac{p_1}{p_2}$

Now, $f_1 = \dfrac{\partial U}{\partial q_1} = MU_1$ and $f_2 = \dfrac{\partial U}{\partial q_2} = MU_2$.

So, $\dfrac{f_1}{f_2} = \dfrac{MU_1}{MU_2}$ = Absolute slope of the indifference curve.

This can be shown in the following manner.

We have, U = f(q_1, q_2)

Taking total derivative, we get, $dU = \dfrac{\partial U}{\partial q_1}.dq_1 + \dfrac{\partial U}{\partial q_2}.dq_2$.

Using different notation, $dU = f_1 dq_1 + f_2\, dq_2$ where $f_1 = \frac{\partial U}{\partial q_1}$ = MU_1 and $f_2 = \frac{\partial U}{\partial q_2}$ = MU_2. Thus, we get, $dU = MU_1 dq_1 + MU_2 dq_2$.

Now, along a given indifference curve, utility is fixed i.e., $dU = 0$

$\therefore MU_1.dq_1 + MU_2 dq_2 = 0$

or, $MU_2\, dq_2 = -\, MU_1 dq_1$

$$\therefore \frac{dq_2}{dq_1} = -\frac{MU_1}{MU_2} = -\frac{f_1}{f_2} < 0.$$

Thus, slope of the indifference curve $\left(\frac{dq_2}{dq_1}\right)$ is negative. Its absolute slope is $-\frac{dq_2}{dq_1} = \frac{MU_1}{MU_2} = -\frac{f_1}{f_2}$. It is called marginal rate of substitution. Thus, MRS = $\frac{MU_1}{MU_2}$.

On the other hand, our budget constraint is :

$p_1 q_1 + p_2 q_2 = M$ or, $p_2 q_2 = -p_1 q_1 + M$

or, $q_2 = -\frac{p_1}{p_2}.q_1 + \frac{M}{p_2}$

This shows that the slope of the budget line is $\left(-\frac{p_1}{p_2}\right)$. So, the absolute slope of the budget line is $\frac{p_1}{p_2}$. Thus, our first order condition for utility maximisation states that,

$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}\left(\text{or,} \frac{f_1}{f_2} = \frac{p_1}{p_2}\right)$ i.e., MRS = $\frac{p_1}{p_2}$

or, slope of indifference curve = slope of budget line.

The second order condition to maximise U requires, $\frac{d^2U}{dq_1^2} < 0$.

We differentiate equation (1) with respect to q_1 and apply the condition

i.e., $\frac{d^2U}{dq_1^2} = f_{11} + f_{12}.\frac{dq_2}{dq_1} + f_{21}\left(-\frac{p_1}{p_2}\right) + f_{22}\frac{dq_2}{dq_1}\left(-\frac{p_1}{p_2}\right) < 0$

or, $f_{11} - f_{12}\left(\frac{p_1}{p_2}\right) - f_{21}\left(\frac{p_1}{p_2}\right) + f_{22}\left(\frac{p_1}{p_2}\right)^2 < 0$

Multiplying both sides by p_2^2, a positive number and putting $f_{21} = f_{12}$(by Young's theorem), we get,

$$f_{11}p_2^2 - 2f_{12}p_1p_2 + f_{22}p_1^2 < 0 \quad ...(a)$$

Let us see the implication of this second order condition. We have, $-\frac{dq_2}{dq_1} = \frac{f_1}{f_2}$

or, $\frac{dq_2}{dq_1} = -\frac{f_1}{f_2}$. Here f_1 (= MU_1) and f_2(= MU_2) both depend on q_1 and q_2.

So, $\frac{dq_2}{dq_1} = -\frac{f_1(q_1,q_2)}{f_2(q_1,q_2)}$. By further differentiation of it with respect to q_1, we get the rate of change of slope of IC.

Now, $\frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^2}\left[f_{11}f_2 + f_{12}\frac{dq_2}{dq_1}.f_2 - f_{21}.f_1 - f_{22}.\frac{dq_2}{dq_1}.f_1\right]$

Putting $\frac{dq_2}{dq_1} = -\frac{f_1}{f_2}$, we get,

$$\frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^2}\left[f_{11}f_2 - f_{12}f_1 - f_{21}f_1 + f_{22}\frac{f_1^2}{f_2}\right]$$

$$= -\frac{1}{f_2^3}\left[f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2\right]$$

Again, we have, $\frac{f_1}{f_2} = \frac{p_1}{p_2}$ or, $f_1 = \frac{p_1}{p_2}.f_2$.

Putting this value, we get, $\frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^3}\left[f_{11}f_2^2 - 2f_{12}f_2^2\frac{p_1}{p_2} + f_{22}f_2^2.\frac{p_1^2}{p_2^2}\right]$

$$= -\frac{1}{f_2.p_2^2}\left[f_{11}p_2^2 - 2f_{12}p_1p_2 + f_{22}p_1^2\right]$$

Inequality (a) shows that the bracketed portion is negative. So, $\frac{d^2q_2}{dq_1^2} > 0$, i.e., the indifference curves are convex from below. Hence inequality (a) implies convexity of indifference curve. Thus, utility maximisation subject to a budget constraint requires fulfilment of two conditions : (i) slope of indifference curve = slope of budget line. (ii) Indifference curve should be convex to the origin.

Example 3.15 : The utility function is : $U = q_1q_2$ and $p_1 = 2$, $p_2 = 5$, $M = 100$. Determine the optimum values of q_1 and q_2 so that utility is maximum.

Solution : Here the budget constraint is : $2q_1 + 5q_2 = 100$. Expressing q_2 as a function of q_1, we get, $5q_2 = 100 - 2q_1$ or, $q_2 = 20 - \frac{2}{5}q_1$

Substituting this into the utility function, $U = q_1q_2 = q_1\left(20 - \frac{2}{5}q_1\right) = 20q_1 - \frac{2}{5}q_1^2$

To maximise U, the first order or the necessary condition is, $\frac{dU}{dq_1} = 0$,

or, $20 - \frac{4}{5}q_1 = 0$, or, $\frac{4}{5}q_1 = 20$ $\therefore\ q_1 = 20 \times \frac{5}{4} = 25$

Then, from the budget constraint, $q_2 = 20 - \frac{2}{5}q_1$

$\therefore\ q_2 = 20 - \frac{2}{5} \times 25 = 20 - 10 = 10$

The second order or the sufficient condition for maximisation of U requires, $\frac{d^2U}{dq_1^2} < 0$. Here, $\frac{d^2U}{dq_1^2} = -\frac{4}{5} < 0$.

So the second order condition is fulfilled. Here, $q_1 = 25$, $q_2 = 10$. Then $U = q_1q_2 = 25 \times 10 = 250$.

Lagrangean method

Suppose we want to maximise $U = f(q_1, q_2)$ subject to a budget constraint $M = p_1q_1 + p_2q_2$. So, it is a problem of constrained maximisation. We form the Lagrangean expression,

$V = f(q_1, q_2) + \lambda(M - p_1q_1 - p_2q_2)$ where λ is the Lagrange multiplier.

It should be noted that here maximisation of U implies maximisation of V as $\lambda(M - p_1q_1 - p_2q_2) = 0$. Further, here $V = V(q_1, q_2, \lambda)$, i.e., V depends on q_1, q_2 and λ. To maximise V, the first order conditions require,

$$\frac{\partial V}{\partial q_1} = 0 \text{, or, } f_1 - \lambda p_1 = 0 \quad \text{or, } f_1 = \lambda p_1 \qquad ...(a)$$

$$\frac{\partial V}{\partial q_2} = 0 \text{, or, } f_2 - \lambda p_2 = 0 \quad \text{or, } f_2 = \lambda p_2 \qquad ...(b)$$

$$\frac{\partial V}{\partial \lambda} = 0 \text{, or, } M - p_1q_1 - p_2q_2 = 0$$

Dividing (a) by (b) we get, $\frac{f_1}{f_2} = \frac{p_1}{p_2}$ or, $\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$ i.e., slope of indifference curve = slope of budget line. The second order condition or the sufficient condition requires that the Hessian Bordered Determinant $|\bar{H}|$ should be positive. [For the concept of Hessian Bordered Determinant, please see Unit 5, section 5].

$$\text{i.e., } [\bar{H}| = \begin{vmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0$$

Expanding the determinant, we get, $-f_{11}p_2^2 + f_{12}p_1p_2 - p_1(-f_{21}p_2 + f_{22}p_1) > 0$

or, $-f_{11}p_2^2 + f_{12}p_1p_2 + f_{21}p_1p_2 - f_{22}p_1^2 > 0$

or, $f_{11}p_2^2 - 2f_{12}p_1p_2 + f_{22}p_1^2 < 0$ (as $f_{12} = f_{21}$).

It implies that $\frac{d^2q_2}{dq_1^2} > 0$. This again implies that indifference curves are convex to the origin.

Example 3.16 : Maximise $U = q_1q_2$ where $p_1 = 2$, $p_2 = 5$ and $M = 100$.

Solution : We have solved this problem by the method of substitution. Now we shall solve the same problem by Lagrangean method. Here our budget equation is : $M = p_1q_1 + p_2q_2$ or, $100 = 2q_1 + 5q_2$. So, we have to maximise $U = q_1q_2$ subject to $100 = 2q_1 + 5q_2$.

We form the Lagrange expression.

$V = q_1q_2 + \lambda(100 - 2q_1 - 5q_2)$. Here, $V = V(q_1, q_2, \lambda)$ and maximisation of V implies maximisation of U as $\lambda(100 - 2q_1 - 5q_2) = 0$. First order conditions to maximise V require,

$$\frac{\partial V}{\partial q_1} = 0 \text{ or, } q_2 - 2\lambda = 0 \text{ or, } q_2 = 2\lambda \quad ...(a)$$

$$\frac{\partial V}{\partial q_2} = 0 \text{ or, } q_1 - 5\lambda = 0 \text{ or, } q_1 = 5\lambda \quad ...(b)$$

$$\frac{\partial V}{\partial \lambda} = 0 \text{, or, } 100 - 2q_1 - 5q_2 = 0 \quad \text{or, } 2q_1 + 5q_2 = 100 \quad ...(c)$$

Dividing (a) by (b), we get, $\frac{q_2}{q_1} = \frac{2}{5}$ or, $2q_1 = 5q_2$. Putting this in (c), we get,

$2q_1 + 2q_1 = 100$ or, $4q = 100$
$\therefore q_1 = 25$, so, $5q_2 = 2 \times 25 \therefore q_2 = 10$

The second order condition or the sufficient condition requires that the Hessian Bordered Determinant $|\overline{H}| > 0$.

$$\text{Here, } |\overline{H}| = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -5 \\ -2 & -5 & 0 \end{vmatrix} = (-1)(-10) - 2(-5) = 10 + 10 = 20 > 0.$$

So, the second order condition is fulfilled.
$\therefore q_1 = 25$, $q_2 = 10$ and $U = 25 \times 10 = 250$

3.10.5 Output Maximisation with Cost Constraint

Let the production function be $q = f(x_1, x_2)$. If r_1 and r_2 are the prices of two inputs X_1 and X_2 respectively then total cost, $C = r_1x_1 + r_2x_2$. We assume that C is fixed at C_0. Then our problem is to maximise $q = f(x_1, x_2)$ subject to the cost constraint $C_0 = r_1x_1 + r_2x_2$. We form the Lagrange expression, $V = f(x_1, x_2) + \lambda(C_0 - r_1x_1 - r_2x_2)$ where λ is the Lagrange multiplier. Here maximisation of V implies maximisation of $q = f(x_1, x_2)$ as $\lambda(C_0 - r_1x_1 - r_2x_2) = 0$. Further, V is now a function of x_1, x_2 and λ, i.e., $V = V(x_1, x_2, \lambda)$. To maximise V, first order conditions are :

$$\frac{\partial V}{\partial x_1} = 0 \text{ or, } f_1 - \lambda r_1 = 0 \text{ or, } f_1 = \lambda r_1 \quad ...(a)$$

$$\frac{\partial V}{\partial x_2} = 0 \text{ or, } f_2 - \lambda r_2 = 0 \text{ or, } f_2 = \lambda r_2 \quad ...(b)$$

$$\frac{\partial V}{\partial \lambda} = 0, \text{ or, } C_0 - r_1x_1 - r_2x_2 = 0 \quad \text{or, } C_0 = r_1x_1 + r_2x_2 \qquad ...(c)$$

Dividing (a) by (b), we get, $\frac{f_1}{f_2} = \frac{r_1}{r_2}$ or, $\frac{MPx_1}{MPx_2} = \frac{r_1}{r_2}$ i.e., slope of isoquant = slope of isocost line. This is our first order condition for output maximisation. The second order condition requires that the Hessian Bordered Determinant $|\overline{H}|$ should be positive, i.e.,

$$|\overline{H}| = \begin{vmatrix} f_{11} & f_{12} & -r_1 \\ f_{21} & f_{22} & -r_2 \\ -r_1 & -r_2 & 0 \end{vmatrix} > 0 \cdot$$

Expanding the determinant, we get, $-f_{11}r_2^2 + f_{12}r_1r_2 - r_1(-f_{21}r_2 + f_{22}r_1) > 0$

or, $f_{11}r_2^2 - 2f_{12}r_1r_2 + f_{22}r_1^2 < 0$ (putting $f_{21} = f_{12}$).

The second order condition may be used to demonstrate that the rate of change of slope of isoquant should be positive, i.e., $\frac{d^2x_2}{dx_1^2} > 0$. This again implies that the isoquant should be convex to the origin. Thus, for output maximisation subject to the cost constraint the conditions are as follows :

(i) **First order or necessary condition :** slope of isoquant = slope of isocost line.

(ii) **Second order or sufficient condition :** isoquant should be convex to the origin.

Example 3.17 : Maximise q = $x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$ when $p_{x_1} = 2$, $p_{x_2} = 4$ and c = 400.

Solution : Here we have to maximise q = $x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$ subject to the cost constraint, 400 = $2x_1 + 4x_2$. So, it is a problem of constrained maximisation. We follow the Lagrange multiplier method. The Lagrange expression is given by :

V = $x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} + \lambda(400 - 2x_1 - 4x_2)$ where λ is the Lagrange multiplier. Here, V = V(x_1, x_2, λ).

First order conditions to maximise V require,

$$V_1 = \frac{\partial V}{\partial x_1} = 0 \text{ or, } \frac{1}{2}.x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}} - 2\lambda = 0, \text{ or, } \frac{1}{2}.x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}} = 2\lambda \qquad ...(a)$$

$$V_2 = \frac{\partial V}{\partial x_2} = 0 \text{ or, } \frac{1}{2}.x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} \quad 4 \quad 0, \text{ or, } \frac{1}{2}.x_1^{\frac{1}{2}}x_2^{\frac{1}{2}} \quad 4 \qquad ...(b)$$

$V_3 = \frac{\partial V}{\partial \lambda} = 0$ or, $400 - 2x_1 - 4x_2 = 0$, or, $2x_1 + 4x_2 = 400$ ...(c)

Dividing (a) by (b) we get, $\frac{\frac{1}{2}x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}}}{\frac{1}{2}x_1^{\frac{1}{2}}x_2^{-\frac{1}{2}}} = \frac{2\lambda}{4\lambda} = \frac{1}{2}$

or, $\frac{x_2}{x_1} = \frac{1}{2}$ or, $x_1 = 2x_2$

Putting this value in equation(c) which is our cost constraint, we get, $2x_1 + 4x_2 = 400$ or, $8x_2 = 400$, $\therefore$ $x_2 = 50$. Then $x_1 = 2x_2 = 100$.

Then output, q = $x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$ = $\sqrt{100 \times 50} = 50\sqrt{2}$. The second order condition requires that the Hessian Bordered Determinant, $|\overline{H}|$ should be positive. That is, $|\overline{H}| > 0$.

Here $|\overline{H}| = \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} > 0$

This condition implies that the iso-quant should be convex to the origin. We assume that the second order condition is fulfilled.

3.10.6 Cost Minimisation with Output Constraint

We shall now try to find out the conditions for cost minimisation subject to a given output. In fact, output maximisation subject a given cost and cost minimisation subject to a given output are just the two sides of the same coin– one implies the other. Hence in both cases, our conditions are same. Let us consider it. Let the production function be $q = f(x_1, x_2)$. Suppose our output is fixed at q_0. So, $q_0 = f(x_1, x_2)$. This is our output constraint. The cost equation of the firm is, $C = r_1x_1 + r_2x_2$. We have to minimise this (objective function) subject to the condition that $q_0 = f(x_1, x_2)$. So, it is a problem of constrained minimisation. We follow the Lagrange technique and form the Lagrange expression : $Z = r_1x_1 + r_2x_2 + \mu[q_0 - f(x_1, x_2)]$ where μ = Lagrange multiplier. Here, $Z = Z(x_1, x_2, \mu)$.

First order conditions to minimise Z require,

$\frac{\partial Z}{\partial x_1} = 0$, or, $r_1 - \mu f_1 = 0$, or, $r_1 = \mu f_1$ ...(a)

$\dfrac{\partial Z}{\partial x_2} = 0$, or, $r_2 - \mu f_2 = 0$, or, $r_2 = \mu f_2$ ...(b)

$\dfrac{\partial Z}{\partial \mu} = 0$, or, $q_0 - f(x_1, x_2) = 0$, or, $f(x_1, x_2) = q_0$ ...(c)

Now, dividing (a) by (b), we get, $\dfrac{f_1}{f_2} = \dfrac{r_1}{r_2}$

i.e., $\dfrac{MPx_1}{MPx_2} = \dfrac{r_1}{r_2}$ or, slope of isoquant = slope of iso-cost line.

The second order condition to minimise Z requires that the Hessian Bordered Determinant should be negative, i.e., $|\overline{H}| < 0$.

So, $|\overline{H}| = \begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -f_1 \\ -\mu f_{21} & -\mu f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} < 0$. Putting, $f_1 = \dfrac{r_1}{\mu}$ and $f_2 = \dfrac{r_2}{\mu}$ from first order conditions (a & b), we get, $\begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -\dfrac{r_1}{\mu} \\ -\mu f_{21} & -\mu f_{22} & -\dfrac{r_2}{\mu} \\ -\dfrac{r_1}{\mu} & \dfrac{-r_2}{\mu} & 0 \end{vmatrix} < 0$. Multiplying the first two columns by $-\dfrac{1}{\mu}$, we get, $^{2}\begin{vmatrix} f_{11} & f_{12} & \dfrac{r_1}{} \\ f_{21} & f_{22} & \dfrac{r_1}{} \\ \dfrac{r_1}{2} & \dfrac{r_2}{2} & 0 \end{vmatrix}$ 0 or, $-\dfrac{1}{\mu}\begin{vmatrix} f_{11} & f_{12} & -r_1 \\ f_{21} & f_{22} & -r_2 \\ -r_1 & -r_2 & 0 \end{vmatrix} < 0$

Since $\mu > 0$, the second order condition for cost minimisation is, $\begin{vmatrix} f_{11} & f_{12} & -r_1 \\ f_{21} & f_{22} & -r_2 \\ -r_1 & -r_2 & 0 \end{vmatrix} > 0$. It implies that the iso-quant should be convex to the origin.

Let us give an example. To show that cost minimisation with output constraint implies output maximisation with cost constraint, we shall take the previous example. We shall just transform the previous example of output maximisation into a case of cost minimsation with output constraint.

Example 3.18 : The cost equation of the firm is : $C = 2x_1 + 4x_2$ and the production function is : $q = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Minimise cost in order to produce $50\sqrt{2}$ units of output.

Solution : Here we have to minimise cost $C = 2x_1 + 4x_2$ (objective function) subject to the condition, $50\sqrt{2} = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ (output constraint). The constraint may be written in an alternative form as, $\left(50\sqrt{2}\right)^2 = \left(\sqrt{x_1 x_2}\right)^2$ or, $5000 = x_1x_2$. This will simplify our differentiation. Thus, our problem formally becomes, Minimise $C = 2x_1 + 4x_2$ subject to the constraint, $5000 = x_1x_2$. Hence, the Lagrangean expression in this case is : $Z = 2x_1 + 4x_2 + \mu(5000 - x_1x_2)$ where μ is the Lagrange multiplier. Here $Z = Z(x_1, x_2, \mu)$.

First order conditions to minimise Z requires.

$$\frac{\partial Z}{\partial x_1} = Z_1 = 0, \text{ or, } 2 - \mu x_2 = 0, \quad \text{or, } \mu x_2 = 2 \qquad ...(a)$$

$$\frac{\partial Z}{\partial x_2} = Z_2 = 0, \text{ or, } 4 - \mu x_1 = 0, \quad \text{or, } \mu x_1 = 4 \qquad ...(b)$$

$$\frac{\partial Z}{\partial \mu} = Z_3 = 0, \text{ or } 5000 - x_1x_2 = 0, \text{ or, } x_1x_2 = 5000 \qquad ...(c)$$

Dividing (a) by (b) we get, $\frac{\mu x_2}{\mu x_1} = \frac{2}{4}$ or, $\frac{x_2}{x_1} = \frac{1}{2}$ or, $x_1 = 2x_2$.

Putting this value of x_1 in equation (c), we get, $x_1x_2 = 5000$, or, $2x_2.x_2 = 5000$, or, $x_2^2 = 2500$, or $x_2 = 50$.

Then $x_1 = 2x_2 = 2 \times 50 = 100$. Then $\mu = \frac{2}{x_2} = \frac{2}{50} = \frac{1}{25}$.

Second order condition to minimise Z requires that the Hessian Bordered Determinant should be negative, i.e., $|\overline{H}| < 0$.

Here, $|\overline{H}| = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}$

Putting the values of the elements of $|\overline{H}|$, we get, $|\overline{H}| = \begin{vmatrix} 0 & -\mu & -x_2 \\ -\mu & 0 & -x_1 \\ -x_2 & -x_1 & 0 \end{vmatrix}$

Expanding, we get, $|\overline{H}| = \mu(-x_1x_2) - x_2(\mu x_1)$

$= -\mu x_1x_2 - \mu x_1x_2 = -2\mu x_1x_2$

$= -\frac{1}{25} \times 50 \times 100 = -200 < 0.$

Thus, the second order condition is fulfilled.

So, $x_1 = 50$, $x_2 = 100$, and the minimum cost to produce $50\sqrt{2}$ units of output is : $C = 2x_1 + 4x_2 = 2(100) + 4(50) = 400$

We should note that in the previous example, the maximum value of output was $50\sqrt{2}$ when C = 400.

3.10.7 Sign of Coefficients of a Cubic Cost Function

Let the cubic cost function be : $C = a_0 + a_1q + a_2q^2 + a_3q^3$. The question is what restrictions should be imposed on the signs of a_0, a_1, a_2 and a_3 so that we can get normal AVC, AC and MC curves. Here, if q = 0, $C = a_0$. So, a_0 = TFC. Hence, $a_0 > 0$. Again, we know that the shapes of MC, AVC and AC are determined by the nature of TVC. Here, TVC $= a_1q + a_2q^2 + a_3q^3$

So, AVC $= \frac{TVC}{q} = a_1 + a_2q + a_3q^2$

To minimise AVC, the first order condition requires,

$\frac{dTVC}{dq} = 0$, or, $a_2 + 2q_3q = 0$ or, $q = -\frac{a_2}{2a_3}$.

The second order condition to minimise AVC requires, $\frac{d^2AVC}{dq^2} > 0$.

Here, $\frac{d^2AVC}{dq^2} = 2a_3$.

So, $\frac{d^2AVC}{dq^2} > 0$, i.e., the second order condition will be fulfilled if $a_3 > 0$.

Again, the AVC-minimising output must be positive. i.e., $-\frac{a_2}{2a_3} > 0$. As $a_3 > 0$, $a_2 < 0$.

Let us consider the minimum value of AVC. We know that AVC is minimum when $q = -\frac{a_2}{2a_3}$. So putting this value in the equation of AVC, we get minimum

$$AVC = a_1 + a_2q + a_3q^2$$

$$= a_1 + a_2\left(-\frac{a_2}{2a_3}\right) + a_3\left(-\frac{a_2}{2a_3}\right)^2$$

$$= a_1 - \frac{a_2^2}{2a_3} + \frac{a_2^2}{4a_3} = a_1 - \frac{a_2^2}{4a_3} = \frac{4a_1a_3 - a_2^2}{4a_3}$$

So, minimum AVC will be positive if $4a_1a_3 - a_2^2 > 0$ or, if $4a_1a_3 > a_2^2$. As $a_3 > 0$, $a_1 > 0$. Thus to get normal U-shaped AVC curve, the restrictions are : (i) $a_0 > 0$, (ii) $a_1 > 0$, (iii) $a_2 < 0$, (iv) $a_3 > 0$ and (v) $4a_1a_3 > a_2^2$. If AVC is u-shaped, then MC and AC will also be u shaped.

We may get similar restrictions on the signs of coeffients taking normal u-shaped MC curve. In that case only the restriction no. (v) will be : $3a_1a_3 > a_2^2$. Readers are requested to check it.

3.11 Summary

1. CONCEPTS OF MAXIMA AND MINIMA OF A SINGLE VARIABLE FUNCTION

When a function attains its highest value it is called maximum value and when it attains its lowest value, it is called minimum value. Separately each one is called extremum and together they are called extrema (extrema is the plural word of extremum). Both maxima and minima are of two types : global and local.

2. IDENTIFICATION OF MAXIMA AND MINIMA

If y = f(x), then its maximisation requires fulfilment of two conditions : (i) First order or necessary condition : $\frac{dy}{dx} = f'(x) = 0$. (ii) Second order or sufficient condition : $\frac{d^2y}{dx^2} = f''(x) < 0$.

Similarly, for minimisation of the function, we require fulfilment of two conditions :

(i) First order or necessary condition : $\frac{dy}{dx} = f'(x) = 0$

(ii) Second-order or sufficient condition : $\frac{d^2y}{dx^2} = f''(x) > 0$

3. POINT OF INFLEXION

Simply speaking, point of inflexion on a curve or function is the point where the curve changes its curvature. If y = f(x), then it will have a point of inflexion if

$\frac{d^2y}{dx^2} = f''(x) = 0$ and $\frac{d^3y}{dx^3}$ $f'''(x) \neq 0$.

4. OPTIMISATION OF MULTIVARIATE FUNCTION

Optimisation is a process or an attempt to achieve an optimum (i.e., maximum or minimum) situation. There may be basically two types of optimisation : unconstrained optimisation and constrained optimisation.

5. UNCONSTRAINED OPTIMISATION

Unconstrained optimisation refers to the process of optimisation of a variable where there is no constraint or condition. If a bivariate function $y = f(x_1, x_2)$ is such that explanatory variables x_1 and x_2 are independent, then we apply unconstrained optimisation, i.e., unconstrained maximisation or unconstrained minimisation of y.

6. CONSTRAINED OPTIMISATION

Constrained optimisation means the maximisation or minimisation of an objective function where the choice variables are not independent : they are subject to some constraint or somehow related to each other. There are two methods of solving a constrained optimisation problem : (i) Method of substitution and (ii) Method of Lagrange multiplier.

7. APPLICATION OF MAXIMA AND MINIMA IN ECONOMICS

The concepts of maxima and minima have numerous applications in Economics. In particular, we may mention the cases of profit maximisation, cost minimisation, output maximisation subject to a given cost, utility maximisation subject to a given budget, etc. They are also used to determine the maximum points of average and marginal product functions, minimum points of marginal cost, average variable cost and average cost functions. In a word, the concepts of maxima and minima are used to determine the optimum level of any decision variable.

3.12 Exercises

Short Answer Type Questions

1. What do you mean by maxima of a single variable function?
2. What is meant by minima of a single variable function?
3. What is an increasing function?
4. What is a decreasing function?
5. What do you mean by a stationary value of a function?
6. What are the conditions for maximisation of a single variable function?
7. State the conditions for minimisation of a single variable function.
8. Mention the necessary and sufficient conditions for maximisation of a function.
9. What are the necessary and sufficient conditions for minimisation of a function?
10. What is meant by point of inflexion of a function?
11. What are the conditions of point of inflexion of a univariate function?
12. What is meant by optimisation of a function?
13. What is unconstrained optimisation?
14. What is meant by constrained optimisation?
15. What are the two methods of solving a constrained optimisation problem?
16. State the relation between AP and MP.

17. State the relation between AC and MC.
18. Let $U = f(q_1, q_2)$. Deduce the slope of an indifference curve.
19. What are the conditions of maximisation of a bivariate function?
20. What are the conditions of minimisation of a bivariate function?
21. When can we apply substitution method in the case of constrained optimisation?
22. When do we apply Lagrangean method in the case of constrained optimisation?

Medium Answer Type Questions

1. Discuss the concepts of maxima and minima of a single variable function.
2. What is global maximum and what is global minimum of a function?
3. Describe the concepts of local maximum and local minimum of a function.
4. How will you identify the maximum and minimum points of a single variable function?
5. Write a short note on the concept of point of inflexion of a single variable function.
6. What are the conditions of unconstrained optimisation of a bivariate function?
7. Discuss the method of substitution in the case of constrained optimisation.
8. Write a short note on sufficient condition for constrained optimisation of a bivariate function.
9. State and mathematically prove the relation between AP and MP.
10. Mathematically prove the relation between AC and MC.
11. How will you determine profit maximising level of employment of labour of a single-product firm?
12. Discuss the conditions for determining profit-maximising sales of a firm under perfect competition.
13. What are the conditions of determining profit maximising output of a firm in an imperfectly competitive market?
14. The cost-function of the firm is $C = q^3 - 3q^2 + 9q$. Determine AC-minimising output.

    Show that at this value of output, MC = AC.
15. Let $C = x^3 - 6x^2 + 15x$ be the total cost function. Show that when AC is minimum, AC = MC.

16. The total cost of the firm is : $C = a_0 + a_1q - a_2q^2 + a_3q^3$. Determine the output when MC is minimum. What is the amount of minimum MC?
17. Let the utility function be : $U \quad aq_1 \quad bq_2 \quad g\sqrt{q_1q_2}$. Determine MRS.
18. The utility function is : $U = \log[(q_1 + a)^{\alpha}(q_2 + b)^{\beta}]$. Determine MRS between q_1 and q_2.
19. Let the production function be : $q = \sqrt{KL}$. Show that MRTS between L and K is given by $\frac{K}{L}$.
20. $C = 100 + 2x + \frac{x^2}{90}$. Calculate minimum AC.
21. Find minimum AC when $AC = 10 - 4x^3 + 3x^4$.
22. Total cost, $C = \frac{1}{3}x^3 - 5x^2 + 75x + 10$. Find minimum MC.

Long Answer Type Questions

1. Discuss the Lagrange multiplier method of constrained maximisation of a bivariate function.
2. Briefly describe the Lagrangean technique of constrained minimisation of a bivariate function.
3. Discuss the conditions of profit maximisation of a firm.
4. How will a firm maximise its output subject to a cost constraint? Mention both necessary and sufficient conditions.
5. Discuss the conditions of utility maximisation subject to the budget constraint of a consumer following substitution method.
6. Analyse how a consumer will maximise utility subject to the budget constraint. Analyse the problem following Lagrangean multiplier method.
7. Deduce the conditions of cost minimisation of a firm subject to an output constraint. Mention both first and second-order conditions.
8. The cost function of the firm is : $C = a_0 + a_1x - a_2x^2 + a_3x^3$ where x = output, a_0, a_1, a_2 and a_3 are positive constants. Determine the value of x at which AVC is minimum. Prove that at this value of x, MC = AVC.
9. Minimise C = 2K + 8L subject to $K^{\frac{1}{2}}L^{\frac{1}{2}} = 8$.
10. Maximise U = x + 2y + xy + 1 subject to 4x + 6y = 130

11. Taking the cost function, $C = a_0 + a_1q - a_2q^2 + a_3q^3$. Show that first MC reaches minimum, then AVC and at last AC.
12. $p = 1200 - 2q$ and $C = q^3 - 61.25q^2 + 1528.5q + 2000$. Determine profit-maximising p and q and also maximum profit.

3.13 References

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Unit 4 ❒ Integration and its Application

Structure

4.1. Objectives

After studying the unit, the readers will be able to know

- the concept of integration and its types
- rules of integration
- various applications of integration in Economics

4.2 Introduction

Integration is a very important tool in mathematical economics. This mathematical concept or tool has so many uses in Economics. In a word, the technique of integration helps us know any total function if its marginal function is given. Further, by applying integration, we can determine the demand function if the value of price elasticity is given, the supply function from the value of elasticity of supply, etc. We can also measure the amounts of consumer's surplus and producer's surplus by using the technique of integration. Hence we discuss about the concept of integration, its types, rules and applications of integration, etc. in this Unit.

4.3 Concept of Integration

The concept of integration may be defined in two alternative ways. **First,** integraion is a process of reverse differentiation. In the process of differentiation, we first take primary function and by differentiation, we reach the derivative function. If we go in the opposite or reverse direction, we go from derivative function to the primary or original function. This reverse process is called integration, or more specifically, indefinite integration and the result obtained through this process is called indefinite integral. In the **second** or alternative sense, integration describes a process of summation. If we want to measure an area enclosed by a curve or a set of curves, we may think of the area consisting of infinite narrow stripes. Summing those stripes we may get the whole area. This process of summation is also called integration, or more specifically, the definite integration and the result obtained through this process of summation is called in definite integral. The process of integration is denoted by the symbol $\int$.

4.4 Indefinite Integral

Let us try to define indefinite integral formally. We have said that integration is the reverse process of differentiation and is denoted by the symbol $\int$. If differentiation of a given function g(x) gives the derivative f(x), we can integrate f(x) to find g(x). Thus, if g(x) is a function of x such that $\frac{d}{dx}[g(x)] = f(x)$, then the indefinite integral of f(x) with respect to x is the function g(x). In notation, $\int f(x)dx = g(x)$. The function f(x) is called the integrand and the function g(x) is called an integral (or, anti-derivative) of the function f(x). For example, since $\frac{d}{dx}(x)^2 = 2x$, $\int 2x\ dx = x^2$.

4.5 Rules of Integration

We now mention some major rules of integration.

Rule 1. **Power rule :** $\int x^n dx = \frac{1}{n+1}.x^{n+1} + C$ where c is constant, $(n \neq -1)$.

Some illustrations :

(i) $\int x^4 dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$

(ii) $\int x\ dx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

(iii) $\int dx = \int 1.dx = \int x^0 dx = \int \frac{x^{0+1}}{0+1} + c = x + c$

Rule 2. **Exponential rule :** $\int e^x dx = e^x + c$ since $\frac{d}{dx}(e^x) = e^x$

(2a) $\int f'(x)e^{f(x)}dx = \frac{f'(x)}{f'(x)}e^{f(x)} + c = e^{f(x)} + c$ since $\int e^{f(x)}dx = \frac{1}{f'(x)}e^{f(x)} + c$

Rule 3. **Logarithmic rule :** $\int \frac{1}{x}dx = \int \frac{dx}{x} = \log x + c$, $(x > 0)$ since $\frac{d}{dx}(\log x) = \frac{1}{x}$

3(a). $\int \frac{f'(x)}{f(x)}dx = \log f(x) + c$

Rule 4. **Integral of a multiple :** $\int kf(x)dx = k\int f(x)dx$ where k is a multiplicative constant. (Note that a variable term cannot be factored out in this fashion).

Illustration : $\int 5x^2 dx = 5\int x^2 dx = \frac{5x^3}{3} + c$

***Rule 5.* Rule of the integral of a sum :** $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$. This can be generalised for any number of sums.

Some Illustrations :

(i) $\int(x^2+x)dx = \int x^2dx + \int xdx = \frac{x^3}{3}+c_1+\frac{x_2}{2}+c_2 = \frac{x^3}{3}+\frac{x^2}{2}+c$

where $c(= c_1 + c_2)$ is a constant.

(ii) $\int(x^3+7x+5)dx = \int x^3dx + 7\int xdx + 5\int dx$

$= \frac{x^4}{4}+c_1+\frac{7x^2}{2}+c_2+5x+c_3$

$= \frac{x^4}{4}+\frac{7x^2}{2}+5x+c$ where $c = c_1 + c_2 + c_3$

***Rule 6.* Rule of substitution :** The integral of $f(u).\frac{du}{dx}$ with respect to the variable x is the integral of f(u) with respect to the variable u. In symbols or notations, $\int f(u)\frac{du}{dx}.dx = \int f(u)du = g(x)+c$.

Illustration : $\int 2x(x^2+1)dx = \int(2x^3+2x)dx = \frac{2x^4}{4}+\frac{2x^2}{2}+c = \frac{x^4}{2}+x^2+c$.

Let us integrate the same expression $2x(x^2 + 1)$ by the rule of substitution.

Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$

$\therefore \frac{du}{2x} = dx$

Now, $\int 2x(x^2+1)dx = \int 2x.u.\frac{du}{2x} = \int u\,du = \frac{u^2}{2}+c_1 = \frac{1}{2}(x^2+1)^2+c_1$

$= \frac{x^4}{2}+x^2+\frac{1}{2}+c_1 = \frac{x^4}{2}+x^2+c$ where $c = \frac{1}{2}+c_1$.

We may integrate in an alternative manner. We have $\int 2x(x^2+1)dx$

$= \int \frac{du}{dx}(x^2 \quad 1)\, dx$ [Putting $x^2 + 1 = u$ and $\frac{du}{dx} = 2x$]

$= \int u\, du = \frac{u^2}{2} + c_1 = \frac{x^4}{2} + x^2 + \frac{1}{2} + c_1 = \frac{x^4}{2} + x^2 + c$ where $c = \frac{1}{2} + c_1$

Rule 7 **: Rule of integration by parts :** The integral of v with respect to u is equal to uv less the integral of u with respect to v.

In notation, $\int v\, du = uv - \int u\, dv$

Let us check it. We know, d(uv) = v du + u dv.

$\therefore \int d(uv) = \int vdu + \int u\, dv \quad \therefore \quad uv = \int vdu + \int udv$. So, $\int vdu = uv - \int udv$

4.6. Definite Integral

The concept of definite integral may be interpreted either as an area or as the limit of a sum. The area enclosed by the curve y = f(x) and the x-axis over a specified domain of x is called the definite integral for the function over this domain. Suppose y = f(x) is a function such that $\int f(x)dx = g(x)$. The definite integral $\int_a^b f(x)dx$ is defined by

$$\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$$

where a and b are two real numbers, and are called the lower and upper limits of the integral, respectively. We give some simple illustrations.

Illustrations : (i) Evaluate $\int_a^b 1.dx$

Ans. $\int 1.dx = \int x^0.dx = x$

So, $\int_a^b 1.du = [x]_a^b = b - a$

(ii) Evaluate $\int_1^5 4x^3 dx$

Ans. $\int_1^5 3x^2 dx = \left[\frac{4x^4}{4}\right]_1^5 = [x^4]_1^5 = 5^4 - 1^4 = 625 - 1 = 624$

(iii) Evaluate $\int_a^b ke^x dx$

Ans. $\int_a^b ke^x dx = [ke^x]_a^b = ke^b - ke^a = k(e^b - e^a)$

4.7 Properties of Definite Integral

Before mentioning the properties of definite integral, we should mention that all functions are not integrable. There are some theorems which specify the conditions under which a function f(x) is integrable. In this connection we may mention the fundamental theorem of calculus. This theorem states that a function y = f(x) is integrable in the interval [a, b] if it is continuous in that interval. The function is then called Reimann integrable.

Having stated the fundamental theorem of calculus, let us mention some important properties of definite integral.

Property 1 : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

For, $\int_a^b f(x)dx = g(b) - g(a) = -[g(a) - g(b)] = -\int_b^a f(x)dx$

Property 2 : A definite integral has a value equal to zero when the two limits of the integration are identical, i.e., $\int_a^a f(x)dx = g(a) - g(a) = 0$

This property can be explained in a very simple manner. Under the 'area' interpretation of definite integral, this means that the area (under a curve)above a single point in the domain is nil. This is quite obvious. On the top of a point on the x-axis, we can draw only a(one dimensional) line, never a (two dimensional) area. The area of a line does not exist.

Property 3 : $^{d}_{a}\, f(x)dx \quad ^{b}_{a}\, f(x)dx \quad ^{c}_{b}\, f(x)dx \quad ^{d}_{c}\, f(x)dx$. This result is also quite obvious. Area under a curve in the interval [a, d] = Area under the curve in the interval [a, b] + area under the interval (b, c) + area under the interval [c, d]. This property is known as the property of additivity. This property can be extended to n sub-intervals.

Property 4 : $\int_a^b -f(x)dx = -\int_a^b f(x)dx$

Property 5 : $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

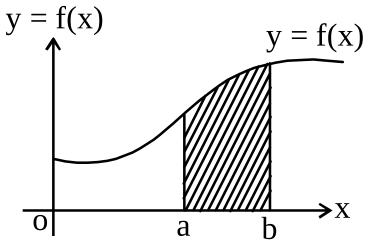
Property 6 : $\int_a^b [f(x)+h(x)]dx = \int_a^b f(x)dx + \int_a^b h(x)dx$

Property 7 : Integration by parts. Suppose there are two functions of x, say, u = u(x) and ν = ν(x).

Then, $\int_{x=a}^{x=b} \nu d\nu = [u\nu]_{x=a}^{x=b} - \int_{x=a}^{x=b} u.d\nu$

4.8 Definite integral as an Area under a Curve

We know that the concept of definite integral can be interpreted as an area. The area enclosed by the curve y = f(x) and the x-axis within an interval of x is called the definite integral for this function over this interval. The idea may be clarified with the help of a diagram. In our figure 4.1 beside we have drawn a continuous function y = f(x). We have also taken two values of x, say, a and b. These are the two limits of x. Here b is the upper limit of x while a is the lower limit of x. Now, definite integral of the function y = f(x) within the interval

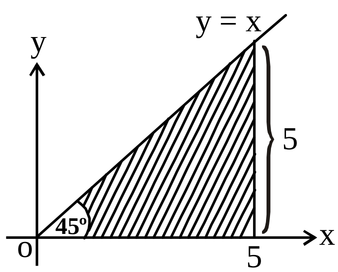


(Fig. 4.1)

[a, b] of x = $\int_a^b f(x)dx$ = g(b) – g(a) = Area under the curve y = f(x) up to x = b minus area under the curve up to x = a. Thus the definite integral may be regarded as an area under a curve. We consider a simple example below.

Example 4.1 : Find the area enclosed by the line y = x, the x-axis and the ordinate at x = 5.

Solution : Here, y = f(x) = x. We have to find out the area of the shaded region shown in figure 4.2. Here the interval of x is [0, 5]. So, we have to calculate definite integral of y = f(x) = x within the interval of [0, 5] of x. Hence, formally,



(Fig. 4.2)

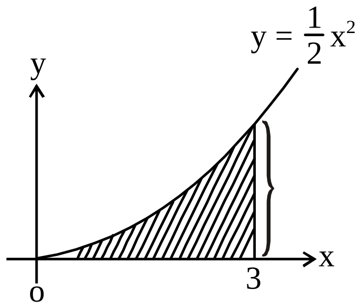
the required area is represented by $\int_0^5 f(x)dx$. Putting f(x) = x, we have the area = $\int_0^5 xdx = \left[\frac{x^2}{2}\right]_0^5 = \frac{5\times5}{2} - 0 = \frac{25}{2} = 12.5.$

In this example we have taken y as a linear function of x. Let us take a non-linear function, say, y = $\frac{1}{2}x^2$. Consider the following example.

Example 4.2 : Find the area enclosed by the curve $\frac{1}{2}x^2$, the x-axis and the ordinate at x = 3.

Ans. Here the required area has been shown by the shaded region in figure 4.3. It is given by :

$$\int_0^3 \frac{1}{2}x^2dx = \left[\frac{x^3}{2\times3}\right]_0^3 = \frac{3\times3\times3}{2\times3} - 0 = \frac{9}{2}$$



(Fig. 4.3)

4.9. Application of Integration in Economics

There are many uses of integration in Economics. We know that integration is the reverse process of differentiation. By differentiating a total function, we can get the marginal function. Hence, by integrating any marginal function we can get the corresponding total function. Thus, we may get the total product (TP) function from the marginal product (MP) function, total revenue (TR) function from the marginal revenue (MR) function, total cost (TC) function from the marginal cost (MC) function, etc. just by applying the technique of integration. We may also derive the demand function from the elasticity of demand, measure the amout of consumer's surphus, volume of producers surplus, etc. by means of integration.We shall consider some of these cases one by one in this section. We first try to find out total functions from the given marginal functions.

4.9.1 Finding out Total Functions from Marginal Functions

Case (i) : Total product (TP) function from marginal product (MP) function.

Let the total product (TP) function be : y = f(L) where y = total product and L = labour. Now, we know that the marginal product of labour is the change in total product due to one unit change in labour employment, *ceteris paribus*. In terms of calculus, $\frac{dTP}{dL} = MP$

So, dTP = MP × dL

Integrating both sides, we get, $\int dTP = \int MP.dL$ or, $TP = \quad MP.dL$

Thus, integrating the MP function, we may get the TP function. Let us give an example.

Example 4.3 : The production function Y = f(L) is such that $\frac{dY}{dL} = 3.\frac{Y}{L}$. Determine the production function.

Solution : We have, $\frac{dY}{dL} = 3.\frac{Y}{L}$ or, $\frac{dY}{Y} = 3.\frac{dL}{L}$

Integrating both sides, we get, $\int \frac{dY}{Y} = 3\int \frac{dL}{L}$

or, log Y = 3 log L + log a where log a is the constant of integration.
or, log Y = log L^3 + log a
or, log Y = log (aL^3)
or, $Y = aL^3$
This is the desired production function in this case.

Case (ii) Total cost (TC) function from marginal cost (MC) function.

Let the total cost function be, C = f(q) where C = total cost and q = output. Then $MC = \frac{dC}{dq}$. So, dC = MC.dq. Integrating we get, $\int dC = \int MC\,dq$ or, $C = \int MC.dq$.

Thus, by applying the technique of integration, we can get the total cost(C) function from the marginal cost (MC) function. Consider the following example.

Example 4.4 : $MC = 500 - 8q + q^2$. If TFC = 6000, determine the total cost (C) function.

Solution : We have, $MC = 500 - 8q + q^2$ or, $\frac{dC}{dq} = 500 - 8q + q^2$

$\therefore\ dC = (500 - 8q + q^2)dq$

Integrating, $\int dC = \int (500 - 8q + q^2)dq = 500\int dq - 8\int q dq + \int q^2 dq$

$= 500q - 8 \times \frac{q^2}{2} + \frac{q^3}{3} + k$ where k is the constant of integration.

So, $C = 500q - 4q^2 + \frac{q^3}{3} + k$.

Now, we are given that TFC = 6000 i.e., if q = 0, C = 6000. Putting q = 0 in our total cost (C) function, we get, k = TFC = 6000. Hence the desired total cost function is :

$C = 500q - 4q^2 + \frac{q^3}{3} + 6000$.

Here 6000 represents the positive vertical intercept of the short run total cost function.

Case (iii) : Total revenue (TR) function from marginal revenue (MR) function.

Let the total revenue (R) function be : R = f(q) where q is the amount of sales of output. So, $MR = \frac{dR}{dq}$ or, dR = MR × dq.

Integrating we get, $\int dR = \int MR.dq$ or, $R = \int MR.dq$

Thus, by integrating the marginal revenue function, we get the total revenue function. Let us give an example.

Example 4.5 : If $MR = 30 - 4q - q^2$, find the TR function.

Solution : We know that total revenue (R) is a function of q, i.e., R = R(q)

Now, $MR = \frac{dR}{dq}$ $\therefore$ dR = MR dq

Integrating, $\int dR = \int MR\,dq = \int (30 - 4q - q^2)dq = 30\int dq - 4\int qdq - \int q^2dq$

or, $R = 30q - 4 \times \frac{q^2}{2} - \frac{q^3}{3} + k$ where k is a constant.

So, $R = 30q - 2q^2 - \frac{q^3}{3} + k$

Now, we know that if q = 0, R = 0 $\therefore$ k = 0

Hence the total revenue function in this case is : $R = 30q - 2q^2 - \frac{q^3}{3}$

Here R = 0 if q = 0. This total revenue function(R) will start from the origin.

Case (iv) : Consumption function from the marginal propensity to consume (MPC).

We assume that consumption(C) is a function of income (Y) i.e., C = f(Y). Then the marginal propensity to consume (MPC) is defined as $\frac{dC}{dY}$. Thus, MPC is the first-order derivative of the consumption function with respect to income, or, MPC is the slope of the consumption function.

Thus, $MPC = \frac{dC}{dY}$ $\therefore$ dC = MPC × dY

Hence, $\int dC = \int MPC.dY$ or, C = $\int MPC.dY$

Thus, integrating the MPC function, we can get the consumption function.

Let us give an example

Example 4.6 : Deduce the consumption function if the marginal propensity to consume (MPC) is $\frac{4}{5}$ and autonomous cousumption is 1000.

Solution : We are given that MPC = $\frac{4}{5}$ or, $\frac{dC}{dY} = \frac{4}{5}$. $\therefore$ $dC = \frac{4}{5}dY$

Integrating both sides, we get, $\int dC = \int \frac{4}{5}dy = \frac{4}{5}\int dY$. $\therefore$ C = $\frac{4}{5}Y + k$

where k is the constant of integration.

Now, it is given that autonomous consumption is 1000, i.e., C = 1000 if Y = 0. Again, from our consumption function, we get, C = k if Y = 0. So, k = 1000. Putting this value of k, we get the desired consumption function : C = $\frac{4}{5}Y + 1000$. Here 1000 represents the positive vertical intercept (i.e., autonomous consumption) of the consumption function.

Case (v) : Saving function from the marginal propensity to save (MPS) function.

We assume that the amount of saving(S) depends on the level of income (Y), i.e., S = S(Y). Then marginal propersity to save is defined as : MPS = $\frac{dS}{dY}$, i.e., MPS is the first order derivative of the saving function.

Now, MPS = $\frac{dS}{dY}$ or, dS = MPS.dY

Integrating we get, $\int dS = \int MPS.dY$ or, $S = \int MPS.dY$

Thus, integrating the MPS function with respect to Y (income), we shall get the saving function.

Consider the following example.

Example 4.7 : $MPS = 0.2 - 0.3Y^{-\frac{1}{2}}$ and when Y = 100, S = 0. Find the saving function.

Solution : We know that if S = S(Y), then $MPS = \frac{dS}{dY}$

$\therefore$ dS = MPS.dY

Integrating, $\int dS = \int MPS.dY$

or, S = $\int (0.2 - 0.3Y^{-\frac{1}{2}})\, dY = 0.2\int dY - 0.3\int y^{-\frac{1}{2}}.dY$

$= 0.2Y - 0.3\frac{Y^{\frac{1}{2}}}{\frac{1}{2}}$ + a where a is the constant of integration.

Thus, S = $0.2Y - 0.6Y^{\frac{1}{2}} + a$.

Now, we are given that S = 0 if Y = 100.

So putting Y = 100, we get, $0.2 \times 100 - 0.6\sqrt{100} + a = 0$

or, 20 – 6 + a = 0 $\therefore$ a = – 14

So, our desired saving function is : S = $0.2Y - 0.6Y^{\frac{1}{2}} - 14$

Here a = –14 is the negative vertical intercept of the saving function.

4.9.2 Demand Function from the Elasticity of Demand

Let the demand function be : q = f(p) where q = quantity demanded and p = price. From the law of demand we know that $\frac{dq}{dp}$ or $f'(p) < 0$ i.e., there is an inverse relation between p and q, *ceteris paribus*. Now, price elasticity of demand may be defined as the proportional change in quantity demanded divided by the proportionate change in price, *ceteris paribus*. Thus, price elasticity of demand, $e_d = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}.\frac{dq}{dp}$.

If the law of demand holds $\frac{dq}{dp} < 0$ and so, $e_d < 0$. The absolute value of price elasticity of demand, $|e_d| = -\dfrac{\frac{dq}{q}}{\frac{dp}{p}}$.

Now, if e_p is given, we can get a relation between $\frac{dq}{q}$ and $\frac{dp}{p}$. Then, by applying the technique of integration, we may get a relation between q(quantity demanded) and p (price). This relation gives us the demand function.

Alternatively, $|e_d| = -\dfrac{d\log q}{d\log p}$ as d log q = $\frac{dq}{p}$ and d log p = $\frac{dp}{p}$. It is the alternative formula of elasticity of demand (e_d) in terms of logarithms. Now, if $|e_d|$ or e_d is given, we can get a relation between d log q and d log p. Then by applying the technique of integration, we may get the relation between q and p. That relation gives us the desired demand function.

We consider two examples showing these two techniques.

Example 4.8 : If $|e_d| = 1$, deduce the demand function.

Solution : $|e_d| = 1$ or $-\dfrac{\frac{dq}{q}}{\frac{dp}{p}} = 1$ or, $\dfrac{dq}{q} = -\dfrac{dp}{p}$.

Integrating, we have, $\int \dfrac{dq}{q} = -\int \dfrac{dp}{p}$

or, log q = –log p + log c where log c is the constant of integration.

Now, log q = $\log\left(\dfrac{c}{p}\right)$

or, q = $\dfrac{c}{p}$ is our demand function or, alternatively, we may write,

log q + log p = log c or, log(pq) = log c

∴ pq = c = constant. This is our demand function. In this case, expenditure of the buyer (pq) is constant and we get a constant outlay curve. Here,the demand curve is a rectangular hyperbola.

Alternative method : We can deduce the same demand function by following a slightly different use of the technique of integration. We are given that $|e_d| = 1$. Using the log-definition of e_d, we can write, $-\frac{d \log q}{d \log p} = 1$ or, d log q = – d log p

Now, integrating, $\int d \log q = -\int d \log p$

or, log q = – log p + log c where log c is a constant.

or, log(pq) = log c $\therefore$ pq = c or, $q = \frac{c}{p}$ is our desired demand function.

We consider another example where the value of price elasticity of demand is a constant not necessarily equal to unity.

Example 4.9 : If the absolute value of price elasticity of demand is α, a constant, deduce the demand function.

Solution : We have, $|e_d| = \alpha$ or, $-\frac{\frac{dq}{q}}{\frac{dp}{p}} = \alpha$ or, $\frac{dq}{q} = -\alpha \frac{dp}{p}$.

Integrating, $\int \frac{dq}{q} = -\alpha \int \frac{dp}{p}$.

or, log q = – αlog p + log a where log a is a constant

$\therefore$ log q = log($ap^{-\alpha}$) $\therefore$ $q = ap^{-\alpha}$. This is our desired demand function.

Alternatively, $|e_d| = \alpha$ or, $-\frac{d \log q}{d \log p} = \alpha$

or, d log q = – α d log p.

Integrating log q = – α log p + log a where log a = constant.

or, log q = log ($ap^{-\alpha}$)

$\therefore$ $q = ap^{-\alpha}$ is the demand function.

Similarly, we can deduce the income-demand function or the Engel function by applying the technique of integration if the value of income elasticity of demand is given. Let the income-demand function or the Engel function be : q = f(M) where q = quantity demanded and M = money income of the buyer. Then the income elasticity of demand is the percentage change in quantity demanded due to one per cent change in money income, *ceteris paribus*. In symbols,

$$e_M = \frac{\frac{dq}{q}\times 100}{\frac{dM}{M}\times 100} = \frac{\frac{dq}{q}}{\frac{dM}{M}} = \frac{M}{q}.\frac{dq}{dM}$$

Now, if e_M is given then we get a relation between $\frac{dq}{q}$ and $\frac{dM}{M}$. Then applying the technique of integration, we shall get a relation between q and M. That relation is nothing but the income-demand function or Engel function.

Alternatively, using the log-definition, we have, $e_M = \frac{d\log q}{d\log M}$.

Again, if e_M is known, we shall have a relation between d log q and d log M. Now, using the technique of integration, we shall get a relation between log q and log M, i.e., between q and M. That relation is our Engel function or the income-demand function.

We consider an example below.

Example 4.10 : If income elasticity of demand, e_M = 1 at all points on the income-demand curve or Engel function, deduce the income-demand curve or the Engel function.

Solution : We have, e_M = 1 or, $\frac{dq/q}{dM/M} = 1$

or, $\frac{dq}{q} = \frac{dM}{M}$. Now, integrating we get, $\int\frac{dq}{q} = \int\frac{dM}{M}$ or, log q = log M + log k where log k = constant.

or, log q = log(kM) $\therefore$ q = kM

This is our income-demand curve or Engel curve which is, in this case, an upward rising straight line passing through the origin.

In this case also, we may follow the alternative method as followed in the case of price elasticity. We first apply the log-definition of elasticity and then use the technique of integration to get the income-demand curve.

We have, e_M = 1

or, $\frac{d\log q}{d\log M} = 1$ or, d log q = d log M

Integrating, $\int d\log q = \int d\log M$

or, log q = log M + log k where log k = constant or, log q = log(kM)

$\therefore$ q = kM is our desired income-demand curve or the Engel curve.

In the same manner we can deduce the income-demand curve if the value of income elasticity is given.

Let $e_M = \beta$, a constant. So, $\frac{dq/q}{dM/M} = \beta$

or, $\frac{dq}{q} = \beta . \frac{dM}{M}$. Now, integrating we get, log q = β log M + log k = log(kM^{β})

$\therefore$ $q = kM^{\beta}$ is our income-demand curve or Engel curve.

Following our alternative method of using log-definition of elasticity, we have, e_M = β.

or, $\frac{d\log q}{d\log M} = \beta \;\; \therefore$ d log q = β.d log M

Integrating, $\int d\log q = \beta \int d\log M$

or, log q = β log M + log k = log(kM^{β})

$\therefore$ $q = kM^{\beta}$ is our income-demand function or Engel function.

4.9.3 Indifference Curve from MRS

By using the technique of integration, we can deduce the equation of the indifference curve if the value of marginal rate of substitution (or the absolute slope of the indifference curve) is given. Let us show it.

Let the utility function of the consumer is : $U = f(q_1, q_2)$. Along an indifference curve (IC), utility level is constant, say, U_0. So, the equation of the IC is : $U_0 = f(q_1, q_2)$. To know the slope of an indifference curve, we take total derivative of the utility function.

Thus, $dU = \frac{\delta U}{\partial q_1}.dq_1 + \frac{\delta U}{\partial q_2}.dq_2 = MU_1.dq_1 + MU_2dq_2$

Now, along a given indifference curve, utility is constant, So, dU = 0. Then we get,

$MU_1dq_1 + MU_2dq_2 = 0$

or, $MU_2dq_2 = -MU_1dq_1$

or, $\frac{dq_2}{dq_1}$ = slope of an indifference curve = $-\frac{MU_1}{MU_2}$.

Under the assumption, $MU_1 > 0$, $MU_2 > 0$, $\frac{dq_2}{dq_1} < 0$ i.e., slope of the indifference curve is negative. The absolute slope of the indifference curve is called the marginal rate of substitution (MRS). Thus, MRS = $-\frac{dq_2}{dq_1} = \frac{MU_1}{MU_2} = \frac{\partial U / \partial q_1}{\partial U / \partial q_2}$.

If this MRS or the absolute slope of the indifference curve is given we can deduce the equation of IC with the help of the technique of integration. We consider an example.

Example 4.11 : The slope of the indifference curve is everywhere equal to $\left(-\frac{q_2}{q_1}\right)$. Deduce the equation of the indifference curve.

Solution : Slope of the indifference curve = $\frac{dq_2}{dq_1} = -\frac{q_2}{q_1}$ or, $\frac{dq_2}{q_2} = -\frac{dq_1}{q_1}$

Integrating, $\int \frac{dq_2}{q_2} = \int \frac{dq_1}{q_1}$ or, $\log q_2 = -\log q_1 + \log U$ where log U is a constant.

$\therefore \log q_2 = \log\left(\frac{U}{q_1}\right)$. So, $q_2 = \frac{U}{q_1}$ or, $U = q_1 q_2$

This is the equation of the indifference curve.

4.9.4 Isoquant from MRTS

By means of the technique of integration we can deduce the isoquant if MRTS is given. Let the equation of the production function be = q = f(K, L) where K and L are the amounts of capital and labour respectively. Along an isoquant, output is fixed, say, q_0, To know the slope of the iso-quant, we take total derivative of the production function.

We get, $dq = \frac{\partial f}{\partial K}.dK + \frac{\partial f}{\partial L}.dL = MP_K dK + MP_L.dL.$

Now, along an iso-quant, output is fixed, say, at q_0. So, $dq_0 = 0$. Then we have, $MP_K dK + MP_L dL = 0$

or, $MP_K dK = -MP_L dL$

So, slope of the isoquant = $\frac{dK}{dL} = -\frac{MP_L}{MP_K}$.

Under the assumption that $MP_K > 0$, $MP_L > 0$, slope of the isoquant = $\frac{dK}{dL} < 0$.

Marginal rate of technical substitution (MRTS) is the absolute slope of the isoquant.

Thus, $MRTS = -\frac{dK}{dL} = \frac{MP_L}{MP_K}$.

Now, if this slope of isoquant or MRTS is given, we can deduce the equation of the isoquant by means of the technique of integration. Consider an example.

Example 4.12 : If MRTS of K and L is given by $\frac{\alpha}{\beta}.\frac{K+b}{L+a}$, deduce the equation of the isoquant.

Solution : We have, $MRTS = -\frac{dK}{dL} = \frac{\alpha}{\beta}.\frac{K+b}{L+a}$

or, $\beta.\frac{1}{K+b}.dK = -\alpha\frac{1}{L+a}.dL$

Integrating we get, $\int \beta\frac{1}{K+b}.dK = -\alpha\int\frac{1}{L+a}.dL$

or, $\beta.\log(K + b) = -\alpha \log (L + a) + \log q$ where log q = constant.

or, $\beta \log(K + b) + \alpha \log(L + a) = \log q$

or, $\log(K + b)^{\beta} + \log(L + a)^{\alpha} = \log q$

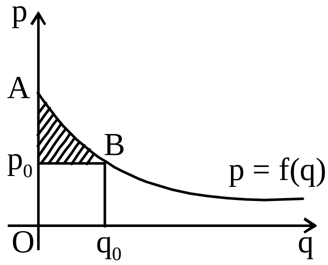
$\therefore \log[(K + b)^{\beta}.(L + a)^{\alpha}] = \log q$

So, $q = (K + b)^{\beta}(L + a)^{\alpha}$

This is the equation of our desired isoquant.

4.9.5 Measurement of Consumer's Surplus

The concept of consumer's surplus has been given by Alfred Marshall. Prof. J. R. Hicks has given a simple but workable definition of consumer's surplus. According to him, consumer's surplus is the difference between the two prices– the price which the consumer is willing to pay rather than go without the thing and the price which he actually pays. In other words, consumer's surplus is the difference between demand price and actual price. We have tried to clarify the concept in figure 4.4. Let our inverse demand function be : p = f(q). Now, at price p_0, the consumer purchases Oq_0 amount of the commodity. So the consumer actually pays = $Op_0 \times Oq_0$ = □$Op_0 Bq_0$. However, the consumer was willing to pay for Oq_0 units = area $OABq_0$. So,



(Fig. 4.4)

consumer's surplus = area $OABq_0$ – area Op_0Bq_0. Hence the triangular are Ap_0B represents consumer's surplus. Thus, formally consumer's surplus = $\int_0^{q_0} f(q).dq - p_0q_0$.

Thus, by applying the technique of integration, we may determine the size of consumer surplus.

Let us give an example on consumer's surplus.

Example 4.13 : Given the inverse demand function, p = 80 – 2q, determine consumer's surplus if p = 30.

Solution : Here p = 80 – 2q = f(q)

If p = 30, then 80 – 2q = 30 or 2q = 80 – 30 = 50 ∴ q = 25

Thus, $p_0 = 30$, $q_0 = 25$.

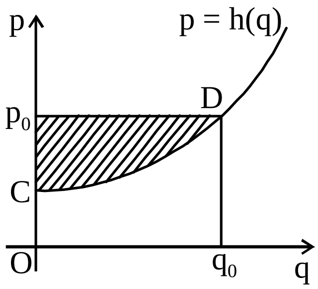
Now, consumer's surplus = $\int_0^{q_0} f(q)\,dq - p_0q_0 = \int_0^{25} (80-2q)dq - 30\times 25$

$= [80q - q^2]_0^{25} - 750 = 80 \times 25 - (25)^2 - 750$

$= 2000 - 625 - 750 = 2000 - 1325 = 675$

4.9.6 Measurement of Producer's Surplus

Producer's surplus is the difference between two prices : price the producer actually receives and the price without which the producer would not sell the commodity. In other words, producer's surplus is the difference between actual price and the minimum supply price. The concept is explained in the figure 4.5.



(Fig. 4.5)

In this figure we have drawn the inverse supply curve p = h(q). Now, at price is p_0, suppose the amount of sale = Oq_0. So, the seller or the producer receives = $Op_0 \times Oq_0$ = area Op_0Dq_0. However, the producer was willing to supply Oq_0 amount of output if he would get the amount = area $OCDq_0$. Thus, producer's surplus = area Op_0Dq_0 – $OCDq_0$. Hence the triangular area Cp_0D respresents producer's surplus. Thus formally producer's surplus

$= p_0q_0 - \int_0^{q_0} h(q)\,dq$

We see that by applying the technique of integration, we can determine the amount of producer's surplus.

Consider the following example.

Example 4.14 : Given the inverse supply function p = 30 + 2q, determine producer's surplus if p = 50.

Solution : We have, p = 30 + 2q

If p = 50, we get, 30 + 2q = 50

or, 2q = 50 – 30 = 20 $\therefore$ q = 10

Thus, we have, $p_0 = 50$, $q_0 = 10$

Now, producer's surplus is given by the expression,

$$= p_0q_0 - \int_0^{q_0} h(q)\,dq = 50 \times 10 - \int_0^{10}(30+2q)dq = 500 - [30q + q^2]_0^{10}$$

$$= 500 - 30 \times 10 - 10^2 = 500 - 300 - 100 = 100$$

4.9.7 Miscellaneous Examples on Application of Integration in Economics

Example 4.15 : MR = a – 4bq. Deduce the demand function.

Solution : We know, MR = $\frac{dR}{dq}$ where R = total revenue. Differentiating total revenue function with respect to q, we get MR function, Hence, integrating MR with respect to q, we shall get total revenue(R) function as integration is the reverse process of differentiation. From this total revenue function, we shall get the average revenue (AR) function or the demand function.

We have, MR = $\frac{dR}{dq}$ $\therefore$ dR = MR.dq

Integrating, $\int dR = \int MR.dq$

or, R = $\int (a - 4bq)dq = aq - \frac{4bq^2}{2} + k$ where k is the constant of integration.

Thus, R = aq – 2bq² + k

Now, we know that if q = 0, R = 0. So, here k = 0. Hence our total revenue function is : $R = aq - 2bq^2$

Now, $R \equiv pq$ $\therefore AR \equiv \frac{R}{q} \equiv \frac{p \times q}{q} = p$.

So, in our case, $AR \equiv p \equiv \frac{R}{q} = \frac{aq - 2bq^2}{q} = a - 2bq$

This is our desired demand function or AR function.

Example 4.16 : If MR = 81 – x^2, find maximum TR. Also deduce the demand function.

Solution : As MR = $\frac{dTR}{dx}$, we can get TR by integrating the MR function.

So, TR = $\int MR\,dx = \int (81 - x^2)dx$ = $81x - \frac{x^3}{3} + k$ where k is a constant.

Now, we know that TR = 0 if x(sales of output) is zero. $\therefore$ k = 0. So, TR = $81x - \frac{x^3}{3}$.

So, AR = $\frac{TR}{x}$ = 81 – $\frac{x^2}{3}$.

Again, TR ≡ p × x, $\therefore$ AR ≡ $\frac{TR}{x} \equiv \frac{px}{x} \equiv p$

So, the demand function is : p (= AR) = 81 – $\frac{x^2}{3}$

Now we shall determine the maximum value of TR by applying the technique of integration. We know that integration of MR gives us TR. So, TR will be maximum when MR = 0. So, we put MR = 0

or, 81 – x^2 = 0, or x^2 = 81 $\therefore$ x = ± 9

As x $\nless$ 0, x = 9

So, when x = 9, TR will be maximum.

Hence we shall integrate MR function with respect to x in the interval [0, 9].

Thus, maximum TR = $\int_0^9 MR\,dx = \int_0^9 (81 - x^2)\,dx = \left[81x - \frac{x^3}{3}\right]_0^9$

= 81 × 9 – $\frac{9 \times 9 \times 9}{3}$ = 81 × 9 – 9 × 9 × 3 = 729 – 243 = 486

We can check our result by applying the technique of differentiation. We have, TR = $81x - \frac{x^3}{3}$. Now, TR will be maximum if (i) $\frac{dTR}{dx} = 0$ and (ii) $\frac{d^2TR}{dx^2} < 0$

Here, from (i), $\frac{dTR}{dx}$ = 81 – x^2.

Putting $\frac{dTR}{dx} = 0$, we get 81 – x^2 = 0 $\therefore$ x = ± 9

Now, $\frac{d^2TR}{dx^2} = -12x$

If x = –9, $\frac{d^2TR}{dx^2} = 18 > 0$. If x = 9, $\frac{d^2TR}{dx^2} = -18 < 0$

So, TR is maximum when x = 9

Now, putting x = 9, maximum TR = $81x - \frac{x^3}{3}$ = 81 × 9 – $\frac{9^3}{3}$ = 486.

Example 4.17 : Given MC = 500 – 6q + q^2, deduce the total cost(TC) function if TFC = 7000 (q = output).

Solution : We know that MC = $\frac{dTC}{dq}$ $\therefore$ dTC = MC dq. So, TC = $\int MCdq$

Putting the equation of MC, we get,

TC = $\int (500 - 6q + q^2)\,dq$

= 500q $- \frac{6q^2}{2} + \frac{q^3}{3}$ + k where k = constant

= 500q – $3q^2$ + $\frac{q^3}{3} + k$

Now, if q = 0, TC = TFC = k, So, k = 7000

Hence the equation of the desired total cost function is :

TC = 500q – $3q^2$ + $\frac{q^3}{3}$ + 7000

Example 4.18 : Deduce the demand function if MR = $\frac{ab}{(q+b)^2} - c$ (a, b and c are constants and q = amount of sale of output).

Solution : We shall first deduce the TR function and then from TR function, we shall get AR function or the demand function.

Now, we know that TR = $\int MRdq$

= $\int \left[\frac{ab}{(q+b)^2} - c\right]dq$ = $\frac{ab(q+b)^{-1}}{-1} - cq + k$ (k = constant)

$$= -\frac{ab}{q+b} - cq + k$$

Further, if q = 0, TR = 0

$\therefore -\frac{ab}{b} + k = 0$ or, k = a

So, TR = $-\frac{ab}{q+b} - cq + a = a - \frac{ab}{q \quad b} \quad cq = \frac{aq + ab - ab}{q+b} - cq$

$\therefore$ TR = $\frac{aq}{q+b} - cq$.

This is our desired TR function.

Now, TR ≡ p × q $\quad \therefore$ AR ≡ $\frac{p \times q}{q} \equiv p$

$\therefore$ p(≡ AR) = $\frac{TR}{q} = \frac{a}{q+b} - c$

This is our desired demand function or the AR function.

Example 4.19 : If MC = AC for all levels of output (q), then prove that total cost(C) is a multiple of q.

Solution : We are given that MC = AC i.e., $\frac{dC}{dq} = \frac{C}{q}$ or, $\frac{dC}{C} = \frac{dq}{q}$

Integrating, $\int \frac{dC}{C} = \int \frac{dq}{q}$

$\therefore$ log C = log q + log m where log m = constant or, log C = log(mq)

So, C = mq i.e., total cost (C) is a multiple (m) of q. In this case, the total cost function is a straight line passing through the origin, and AC = $\frac{C}{q}$ = m and MC = $\frac{dC}{dq}$ = m .

So, AC = MC = m will be a horizontal straight line and they will coincide.

Example 4.20 : If the elasticity of factor substitution (σ) = 1, deduce the production function.

Solution : We are given that the elasiticity of substitution,

$$\sigma = 1,\ \sigma = \frac{\dfrac{d(K/L)}{K/L}}{\dfrac{d(MP_L/MP_K)}{MP_L/MP_K}}$$

In terms of logarithms,the formula of elasticity of substitution, $\sigma = \dfrac{d\log(K/L)}{d\log(MP_L/MP_K)}$

Now, $\sigma = 1$. So, d log (K/L) = d log(MP_L/MP_K)

Integrating both sides, we get,

log(K/L) = log(MP_L/MP_K) + log(α/β) where log (α/β) = constant

or, log (K/L) = $\log\left(\dfrac{\alpha}{\beta}.\dfrac{MP_L}{MP_K}\right)$ or, $\dfrac{K}{L} = \dfrac{\alpha}{\beta}.\dfrac{MP_L}{MP_K}$

Now, $\dfrac{MP_L}{MP_K}$ is the absolute slope of the isoquant, i.e., $\dfrac{MP_L}{MP_K} = -\dfrac{dK}{dL}$. Putting this value, we get, $\dfrac{K}{L} = -\dfrac{\alpha}{\beta}\dfrac{dK}{dL}$ or, $\beta.\dfrac{dL}{L} = -\alpha.\dfrac{dK}{K}$

Again integrating, we get, $\beta.\log L = -\alpha \log K + \log\left(\dfrac{q}{A}\right)$

or $\beta.\log L + \alpha \log K = \log\left(\dfrac{q}{A}\right)$ or, $\log(K^{\alpha}L^{\beta}) = \log\left(\dfrac{q}{A}\right)$ where log $\left(\dfrac{q}{A}\right)$ is a constant.

So, $\dfrac{q}{A} = (K^{\alpha}L^{\beta})$, or, $q = AK^{\alpha}L^{\beta}$

This is our desired production function. In this case, our production function is Cobb-Douglas type. We know that in the case of Cobb-Douglas production function, the elasticity of substitution is equal to unity.

Example 4.21 : Given the demand function p = 20 – 3x, find consumer's surplus assuming that market equilibrium is attained at p_0 = 5, x_0 = 5.

Solution : Consumer's surplus $= \int_0^5 (20-3x)\,dx - p_0x_0$

$$= \left[20x - \frac{3x^2}{2}\right]_0^5 - 5\times5 = 20 \times 5 - \frac{3\times5\times5}{2} - 25$$

$$= 100 - \frac{75}{2} - 25 = \frac{200-75-50}{2} = \frac{75}{2} = 37.5$$

Example 4.22 : The demand law for a commodity is : p = 20 – 2D – D^2. Find consumer's surplus when demand (D) is 3.

Solution : When D = 3, p = 20 – 2D – D^2 = 20 – 2 × 3 – 3^2 = 20 – 15 = 5.

So, p_0 = 5 and D_0 = 3

Now, consumer's surplus = $\int_0^{D_0} f(D)\,dD - p_0D_0 = \int_0^3 (20-2D-D^2)\,dD - p_0D_0$

$$= \left[20D - D^2 - \frac{D^3}{3}\right]_0^3 - 3\times5 = 20 \times 3 - 3^2 - \frac{3^3}{3} - 15$$

= 60 – 9 – 9 – 15 = 60 – 33 = 27

Example 4.23 : Demand function for a commodity is : p = 20 – 3D. The supply function on this market is : p = 2D. Find consumer's surplus at equilibrium price. Also find producer's surplus at the equilibrium point.

Putting demand price = supply price, we get, 2D = 20 – 3D, or 5D = 20 ∴ D = 4,

Then p = 2D = 2 × 4 = 8. Thus, p_0 = 8, D_0 = 4.

Now, consumer's surplus = $\int_0^4 (20-3D)dD - p_0D_0$

$$= \left[20D - \frac{3}{2}D^2\right]_0^4 - 8\times4 = 20 \times 4 - \frac{3}{2}\times4^2 - 32 = 80 - 3 \times 8 - 32 = 80 - 56 = 24$$

Producer's surplus will be obtained by utilising the supply function. Here, producer's suplus at the equilibrium combination $p_0 = 8$ and $D_0 = 4$,

$$= p_0D_0 - \int_0^4 2D\,dD = 8 \times 4 - \left[2.\frac{D^2}{2}\right]_0^4 = 8 \times 4 - \left[D^2\right]_0^4 = 8 \times 4 - 4 \times 4 = 16$$

Example 4.24 : Given the demand function, $p_d = 4 - x^2$ and the supply function $p_s = x + 2$, find consumer's surplus and producer's surplus assuming perfect competition.

Solution : Putting $p_d = p_s$, we get, $4 - x^2 = x + 2$

or, $x^2 + x - 2 = 0$ or, $(x + 2)(x - 1) = 0$

$\therefore$ $x = -2, 1$. As $x \nless 0$, $x = 1$

Then $p = x + 2 = 1 + 2 = 3$. Thus, $p_0 = 3$, $x_0 = 1$

Now, consumer's surplus $= \int_0^1 (4 - x^2)dx - p_0x_0 = \left[4x - \frac{x^3}{3}\right]_0^1 - 3 \times 1$

$$= 4 - \frac{1}{3} - 3 = \frac{12 - 1 - 9}{3} = \frac{2}{3}$$

Producer's surplus $= p_0x_0 - \int_0^1 (x + 2)dx$

$$= 3 \times 1 - \left[\frac{x^2}{2} + 2x\right]_0^1 = 3 - \frac{1}{2} - 2 = \frac{6 - 1 - 4}{2} = \frac{1}{2}$$

It may be noted that to determine consumer's surplus, we have used the demand function i.e., the equation of demand price (p_d) while, to determine producer's surplus, we have used the supply function i.e., the equation of the supply price (p_s).

Example 4.25 : The demand function is : $D = \frac{25}{4} - \frac{p}{8}$ while the supply function is $p = 5 + D$. Determine consumer's surplus and producer's suplus at equilibrium price.

Solution : To determine equilibrium price, we first put the value of D from the demand function into the supply function.

Thus $p = 5 + D = 5 + \frac{25}{4} - \frac{p}{8}$

or, $p + \frac{p}{8} = 5 + \frac{25}{4}$ or, $\frac{9p}{8} = \frac{45}{4}$ $\therefore$ $p = \frac{45}{4} \times \frac{8}{9} = 10$

Now, p = 5 + D $\therefore$ D = p – 5 = 10 – 5 = 5

Thus, equilibrium price = p_0 = 10 and equilibrium quantity = D_0 = 5.

Again, our demand function is : $D = \frac{25}{4} - \frac{p}{8}$

Writing it in inverse form (i.e., p as a function of D) we get, $\frac{p}{8} = \frac{25}{4} - D$

or, p = 50 – 8D.

This is the inverse demand function.

Now, consumer's surplus at equilibrium price and quantity,

$$= \int_0^5 (50 - 8D)dD - p_0D_0 = \left[50D - \frac{8}{2}D^2\right]_0^5 - 10 \times 5$$

$= 50 \times 5 - 4 \times 5^2 - 10 \times 5 = 250 - 100 - 50 = 100$

Now, producer's surplus at p_0 = 10 and D_0 = 5 will be obtained by utilising the supply function. Here the supply function is given as the inverse supply function i.e., price as the function of supply.

Now, producer's surplus = $p_0D_0 - \int_0^5 (5 + D)dD$

$$= p_0D_0 - \left[5D + \frac{D^2}{2}\right]_0^5 = 10 \times 5 - 5 \times 5 - \frac{5 \times 5}{2}$$

$$= 50 - 25 - \frac{25}{2} = \frac{100 - 50 - 25}{2} = \frac{25}{2} = 12.5$$

4.10 Summary

1. Concept of Integration : The mathematical technique of integration has two meanings. **First**, integration means a process of reverse differentiation. It is more specifically called indifinite integration. In the **second** or alternative meaning, integration means a process of summation. More specifically, it is called definite integration. The result of integration is called integral. The function or which the technique of integration is applied, is called integrand. The process of integration is denoted by the symbol $\int$. There are some rules of integration.

2. Indefinite Integral : If g(x) is a function of x such that $\frac{d}{dx}[g(x)] = f(x)$, then the infinite integral of f(x) with respect to x is the function g(x). In notation, $\int f(x)dx = g(x)$.

3. Definite Integral : Definite integral may be regarded either as an area or as the limit of a sum. The area enclosed by the curve y = f(x) and the x-axis over an interval of x is called the definite integral for the function over that interval. Definite integral has some important properties.

4. Application of Integration in Economics : There are many uses of integration in Economics. As a reverse process of differentiation, integration helps us to know the total function from its marginal function. Thus, by applying the technique of integration, we can get the total pruduct function from the marginal product function, total revenue function from the marginal revenue function, total cost function from the marginal cost function, indifference curve from its slope or marginal rate of substitution (MRS), isoquant from its slope or marginal rate of technical substitution (MRTS), etc. We can also derive the demand function by means of integration if the elasticity of demand is given. Again, as a measure of area under a curve, integration may be used to measure consumer's surplus, producer's surplus, etc.

4.11 Exercises

Short Answer Type Questions

1. What is integration?
2. Define definite integral.
3. What do you mean by indefinite integral?
4. State the power rule of integration and give an example.
5. Evaluate $\int 5x^4dx$
6. Evaluate $\int dx$
7. State the exponential rule of integration.
8. State the logarithmic rule of integration.

9. What is the rule of integral of a multiple?
10. State the rule of the integral of a sum.
11. Evaluate $\int (x^3 + 3x + 5)\,dx$
12. Evaluate $\int (5x^2 - 7x - 8)\,dx$
13. State the rule of substitution used in integration.
14. Evaluate $\int_1^3 7x^2 dx$
15. Evaluate $\int_c^d 1.dx$
16. Evaluate $\int_p^q 7e^x dx$
17. State the fundamental theorem of calculus.
18. Define consumer's surplus in terms of integration.
19. Define producer's surplus using the concept of integration.
20. If income elasticity of demand, $e_M = 1$ at all points on the income demand function, then deduce the income-demand function or the Engel function.

Medium Answer Type Questions

1. Distinguish between definite integral and indefinite integral.
2. State the rule of integration by parts.
3. Illustrate the rule of substitution with a suitable example.
4. Illucidate the concept of integration as an area.
5. How can you get TR function from MR function and TC function from MC function?
6. How will you get the demand function by the application of integration, from the elasticity demand?
7. If the absolute value of price elasticity of demand is β, then deduce the demand function.

8. The slope of indifference curve $\left(\frac{dy}{dx}\right)$ is everywhere equl to $\left(-\frac{y}{x}\right)$. Deduce the equation of the indifference curve.

9. $MC = 2 + 3\sqrt{q} + \frac{5}{\sqrt{q}}$. Find TC if f(1) = 21

10. MC = 25 + 30q – $9q^2$ and TFC = 100. Find TC, TVC, AC and AVC.

11. MPC = 4/5 and C = 100 when Y = 0. Deduce the consumption function.

12. p = 45 – $\frac{q}{2}$ is the demand function. Find consumer's surplus if p = 32.5.

13. $p_a = 4 - q^2$ and $p_s = q + 2$. Determine consumer's surplus and producer's surplus under perfect competition.

14. $p_d = 16 - q^2$ and $p_s = 2q^2 + 4$. Determine consumer's surplus and producer's surplus in equilibrium.

Long Answer Type Questions

1. State the rules of integration with suitable illustrations.

2. State the basic properties of definite integral.

3. Mention some major applications of integration in Economics.

4. (a) MR = 15 – 2q – q^2. Find maximum TR.

   (b) MR = a – 2bq. Derive TR and AR functions.

5. (a) MC = 2 + $3e^q$. Find TC if TFC = 500.

   (b) If MC = a + bq, deduce AC function.

6. (a) MPC = $\frac{0.4}{\sqrt{Y}}$ and C = 80 when Y = 0. Deduce the consumption function .

   (b) MPS = $0.3 - 0.1Y^{-\frac{1}{2}}$ and S = 0 when Y = 81. Find the saving function.

7. MRS = $\frac{q_1 - a}{q_2 - b}$. Show that one form of the utility function is :

   $U = (q_1 - a)^2 + (q_2 - b)^2$

8. Let MRS = $\frac{\alpha}{\beta}.\frac{q_2+b}{q_1+a}$. Show that one form of the utility function is :

$U = (q_1 + a)^{\alpha}(q_2 + b)^{\beta}$.

4.12 References

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Unit 5 ❒ Matrices and Determinants

Structure

5.1 Objectives

After studying the unit, the reader will be able to know

- Matrix and its different types
- Matrix operations
- Determinant and its properties
- Matrix inversion and its application to solve simultaneous equations
- Concepts of Hessian Determinant and Hessian Bordered Determinant
- Applications of matrix and determinant in Economics

5.2 Introduction

In economic models, we have, in many cases, a set of simultaneous equations. We are required to solve those simultaneous equations. Matrices and determinants greatly help us in this regard. Further, matrices are often used to simplify notation when dealing with a large number of simultaneous equations. Hence we shall consider in this unit the concepts of matrices and determinants and their uses in solving simultaneous equations very often used in economic models.

5.3 Definition and Concept of a Matrix

Any rectangular array of numbers is called a matrix. A matrix with m rows and n columns is of the order (m × n). An (m × 1) matrix is called a column vector and a (1 × m) matrix is a row vector. The terms 'array' and matrix are used interchangeably. If A denotes the array or the matrix of order (m × n) then the matrix A may be written as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ or, } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Thus, a matrix is denoted by the symbol [ ] or ( ). The elements of the matrix A of order (m × n) are the coefficients a_{ij}(i = 1, 2, ..., m, j = 1, 2, ..., n) where the first subsctipt is the row index and the second subscript is the column index. Thus, a_{ij} is the element of matrix A in i-th row and j-th column. For example, a_{37} is the element of a matrix or of an array in its third row and seventh column.

5.4 Matrix Operations

We have said that a matrix is any rectangular array of numbers (real or complex). To deal with matrices, we have to know matrix operations. We here mention below some basic matrix operations.

(i) Addition of matrices : To add two or more matrices, they should be comfortable for addition. Two or more matrices are said to be comfortable for addition, if and only if, they are of the same order. Then, if $A = (A_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, then the sum of these two matrices, A + B is defined by the matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$. Thus, by adding the corresponding elements of two or more matrices, we can get the sum of those two or more matrices. We give an example.

Example 5.1. : $A = \begin{bmatrix} 3 & 7 \\ 9 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$. Find A + B.

Solution : Let C = A + B. So, $C = \begin{bmatrix} 3 & 7 \\ 9 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3+5 & 7+3 \\ 9+2 & 4+6 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 11 & 10 \end{bmatrix}$

This addition operation of matrices will hold for any number of matrices, provided they are comfortable for addition.

(ii) Subtraction of matrices : Subraction is also one kind of addition and hence the operation of subtraction of matrices is exactly similar to that of addition. Now, two or more matrices are comfortable for subtraction if they are of the same order. If $A = (a_i)_{m\times n}$ and $B = (b_i)_{m\times n}$, then the difference A – B is defined to be the sum of A + (–B). If this difference is defined by the matrix C, then $C = (c_{ij})$ where $c_{ij} = a_{ij} - b_{ij}$. Thus, simply by subtracting the corresponding elements of two matrices, we may get their difference. We cite an example.

Example 5.2. : If $A = \begin{bmatrix} 10 & 5 & 8 \\ 7 & 9 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 & 9 \\ 5 & 3 & 6 \end{bmatrix}$

Solution : Let C = A – B

Then $C = A - B = \begin{bmatrix} 10 & 5 & 8 \\ 7 & 9 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 7 & 9 \\ 5 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 10-6 & 5-7 & 8-9 \\ 7-5 & 9-3 & 2-6 \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 & -1 \\ 2 & 6 & -4 \end{bmatrix}$

(iii) Product of matrices : Multiplication of two matrices is possible if the two matrices are comfortable for the product. When the number of columns of a matrix A is the same as the number of rows of another matrix B, then A is said to be comfortable for the product AB. We then say that the product AB is defined and it is denoted by A.B or AB.

Suppose, A = (a_{ij}); where i = 1, 2, ... m, j = 1, 2, ..., n, i.e., A is a matrix of order (m × n). Further, B = (b_{jk}) where j = 1, 2, ..., n, k = 1, 2, ..., p, i.e., B is a matrix of order (n × p). Here, the number of columns in A = the number of rows in B = n. So, the product AB is defined and it will be a matrix of order (m × p). Let the product be the matrix C. Then, C = (c_{ik}) where $c_{ik} = a_{il}b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$.

Here C is a matrix of order (m × p). Let us give an example.

Example 5.3 : Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$

Find AB. Also finf BA; if possible.

Solution : Here A is a (2 × 3) matrix and B is a (3 × 2) matrix. So, the number of columns of matrix A = the number of rows of matrix B = 3. So, the product AB is defined or A is comfortable for the product AB. Here AB will be of order (2 × 2).

$$\text{Here, } AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2+3\times3 & 1\times0+2\times4+3\times4 \\ 0\times1+4\times2+5\times3 & 0\times0+2\times4+4\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 20 \\ 23 & 28 \end{bmatrix}$$

Let us consider BA. We have, $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}_{3\times2}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}_{2\times3}$

Here B is of order (3 × 2) while A is of order(2 × 3). Thus, the number of columns of matrix B = number of rows of matrix A = 2, So, we can find BA or B is comfortable for the product BA. Here the product BA will be of order(3 × 3)

$$\text{Here BA} = C = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1\times1+0\times0 & 1\times2+0\times4 & 1\times3+0\times5 \\ 2\times1+2\times0 & 2\times2+2\times4 & 2\times3+2\times5 \\ 3\times1+4\times0 & 3\times2+4\times4 & 3\times3+4\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & 16 \\ 3 & 22 & 29 \end{bmatrix}$$

We see that BA is of the order (3 × 3). It may be noted that in this case, AB ≠ BA.

5.5 Different Types of Matrices

There are different types of matrices. We here mention some major types.

(i) Column matrix : We have said that a vector is a special type matrix with a single row or single column. Thus, a matrix with a single column is called a column matrix or a column vector. Its order is (m × 1). For example, $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_m \end{bmatrix}$ is a column matrix or a column vector of order (m × 1). Taking a specific numerical example, $A = \begin{bmatrix} 3 \\ 7 \\ 4 \\ 1 \end{bmatrix}$ is a column matrix or column vector of order (4 × 1). In both cases, the number of column is one.

2. Row matrix : A matrix having a single row is called a row matrix or a now vector. Its order is (1 × n). Thus, A = $[a_1\ a_2 \dots a_n]$ is a row matrix or ros vector of order (1 × n). Taking numerical example, A = [3 0 7 5 –2] is a row matrix or row vector of order (1 × 5). In both examples, the number of row is one.

3. Transposed matrix : If rows and columns of a matrix are interchanged, then the new matrix thus obtained is called transposed matrix of the original matrix. If A is the original matrix, then the transpose of matrix A is denoted by A′ or A^T.

Thus, if $A = (a_{ij})$ of order $(m \times n)$, then transpose of A, denoted by, A' or $A^T = (a'_{ij})$ of order $(n \times m) = (a_{ji})$ of order $(n \times m)$.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$, then A' or $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}_{3\times 2}$

Taking a numerical example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$, then A' or $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

We have some properties of transpose of a matrix.

Property 1 : Transpose of transpose of a matrix is the original matrix, i.e., $(A')' = A$

Example : Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$. Then $A' = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$

Now, $(A')' = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} = A$

Property 2 : Transpose of the sum (or difference of matrices is the sum (or difference) of the transposes of the individual matrices, i.e., $(A \pm B)' = A' \pm B'$.

Example : Let $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

Then, $A + B = \begin{bmatrix} 4 & 8 \\ 7 & 11 \end{bmatrix}$ and $(A + B)' = \begin{bmatrix} 4 & 7 \\ 8 & 11 \end{bmatrix}$

Again, $A' = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$. Then $A' + B' = \begin{bmatrix} 4 & 7 \\ 8 & 11 \end{bmatrix} = (A + B)'$.

Similarly it can be checked that $(A - B)' = A' - B'$.

In our example, $(A - B) = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ and $(A - B)' = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Again, we have, $A' = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

$\therefore A' - B' = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = (A - B)'$

(iv) Square matrix : A matrix having equal number of rows and columns is called a square matrix.

Thus, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is a square matrix of order n. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then A is a square matrix of order 2. Similarly, $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3. In our example 5.1, all 3 matrices A, B and C were square matrices of order 2, but in example 5.2, none of the 3 matrices A, B and C was a square matrix.

(v) Symmetric matrix : A square matrix (a_{ij}) for which ij = ji for all i and all j is called a symmetric matrix. Fow example, suppose $A = (a_{ij})$ where i = 1, 2, 3 and j = 1, 2, 3. Thus, A $= \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$. Now, A will be called a symmetric matrix if $a_{12} = a_{21}$, $a_{23} = a_{32}$ and $a_{13} = a_{31}$. We give an example putting values for a_{ij}'s.

Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Then if we interchange a_{ij}'s for a_{ji}'s, i.e., if we take the transpose of A, then

$$A' = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A.$$

Hence, A is a symmetric matrix. Thus, in case of a symmetric matrix, A = A′.

Thus the see that transpose of a symmetric matrix is the original matrix itself, i.e., if A is a symmetric matrix, then A′ = A. The converse is also true. i.e., if A′ = A, then A is a symmetric matrix.

(vi) Diagonal matrix : A diagonal matrix is a square matrix with all its non-diagonal elements equal to zero. Thus,

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$ is a diagonal matrix of order n. Taking a numerical example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3. If may be noted that a diagonal matrix is a symmetric matrix also i.e., A′ = A for a diagonal matrix.

(vii) Identity matrix or unit matrix : It is a special case of a diagonal matrix. If all the diagonal elements of a diagonal matrix are equal to 1, then the matrix is called a unit matrix or an identity matrix. It is generally denoted by the symbol I. Thus $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3, and so on.

It may be noted that for an identity matrix, I = I′. Pre-multiplying or post-multiplying any comfortable matrix A by it gives the same matrix. For example, if A and I are comfortable, and I and A are also comfortable, then AI = IA. This will happen when A and I are square matrices.

(viii) Orthogonal matrix : A square matrix A is said to be an orthogonal matrix if A′A = AA′ = I.

(ix) Null matrix : A matrix with all elements equal to zero is called a null matrix. A null matrix is denoted by 0. Thus, $0=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ or $0=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, etc. are examples of null matrices. A null matrix is also called a zero matrix. Clearly, for null matrices, we have, $0_{p\times q}A_{q\times r}=0_{p\times r}$ and $A_{p\times q}\pm 0_{p\times q}=A_{p\times q}$.

(x) Idempotent matrix : An idempotent matrix is a symmetric matrix which produces itself when it is multiplied by itself. Thus, a symmetric matrix A will be termed as idempotent if AA = A.

It may be noted that the identity matrix I is an idempotent matrix i.e., II = I. Let us check it. We have, I = $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Now, II = $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ = I

Thus, I is an idempotent matrix.

We should also note that as I is symmetric, I′ = I.

5.7 Determinant of a Matrix and its Associated Concepts

In general, a determinant is a square array of numbers. It is so called as it is used in the determination of the solution of a system of simultaneous equations. To every square matrix A = (a_{ij}) of order n, there corresponds a number known as the determinant of matrix A. It is denoted by |A| or Δ or $\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$. The order of a determinant is the number of its rows (or its columns since the array is square). In our above example, the given determinant is of order n.

Let us see how determinants help in the determination of the solution of a system of simultaneous equations. We take the simplest case where we have a system with two equations in two unknowns, x and y.

$a_1x + b_1y = k_1$

$a_2x + b_2y = k_2$

where a_1, a_2, b_1, b_2, k_1 and k_2 are known constants. By the process of elimination we can very simply solve this system for x and y. This process gives,

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{k_2a_1 - k_1a_2}{a_1b_2 - a_2b_1}.$$

It may be noted that the denominator is the same in both expressions and its is computed from products of the co-efficients of x and y. We may write the coefficients in an array $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

From this, it is obvious that the denominator can be obtained by taking the product indicated by the downward sloping arrow (a_1b_2) and then subtracting from it the product indicated by the upward sloping arrow (a_2b_1). Similarly, we may write the numerators as arrays of coefficients as follows :

For x , $\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix} = k_1b_2 - k_2b_1$, and for y, $\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix} = a_1k_2 - a_2k_1$

Thus, the solution of above system may be written in the form of following arrays :

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

We refer to arrays of this kind as determinants, since they help in the determination of the solution of a system of simultaneous equations.

In this connection, we like to mention that a determinant is, by definition, a scalar. However, a matrix does not have a numerical value. In other words, a determinant is reducible to a number, however a matrix is, in contrast, a whole block of numbers. Further, a determinant is defined only for a square matrix while a matrix as such need not be square.

Expansion of a determinant : Let us see how we can expand a determinant. Expansion of a determinant is the computation of its value. Suppose we have the following determinant of order n.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

Here, any element a_{ij} is the element in the ith row and the jth column of the determinant (i, j = 1, 2, 3, ..., n). Thus a_{34} is the element in the third row and fourth cloumn. Let us see how we can expand this determinant or compute its value. Before doing that, we give the following essential definitions.

Principal diagonal : The principal diagonal of a determinant consists of the elements in the determinant which lie in a straight line from upper left-hand corner to the lower right-hand corner. In our above determinant of order n, the elements of its principal diagonal are a_{11}, a_{22}, ..., a_{33}, ..., a_{nn}.

Minor : The minor of an element belonging to a determinant of order n is the determinant of order (n – 1) obtained by deleting the row and column which contain the particular element. For example, in the following fourth order determinant, the minor of the element a_{23} can be obtained by removing the row and column containing this element, i.e., by removing the second row and the third column of the original determinant.

$$\text{Original determinant} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$ and it is of order (4 – 1) = order 3. Generally, the minor of an element is denoted by a capital letter with the same subscrips of the given element. Thus, in symbol, the minor of $a_{23} = A_{23}$.

Cofactor : The cofactor of an element belonging to a determinant is its minor preceded by a '+' or '–' sign according as the sum of the subscripts of the element is even or odd. For example, the cofactor of the element $a_{23} = -A_{23}$ as 2 + 3 = 5, an odd number. Similarly, cofactor of $a_{33} = +A_{33}$ as 3 + 3 = 6, an even number.

The value or the expansion of a determinant may be obtained by using the cofactors (pre-signed minors) of its elements. The steps are given below :

Step 1 : Choose any row (or column). (To avoid any confusion students are advised to choose always the first row).

Step 2 : Multiply each element in the chosen row (or column) by its cofactor.

Step 3 : Add algebraically the products obtained in step 2.

For example, expanding the determinant, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we get,

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}A_{11} + a_{12}(-1)A_{12} + a_{13}A_{13}$$

For example, let us expand the determinant,

$$\Delta = \begin{vmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ 7 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 4(-1)\begin{vmatrix} 2 & 3 \\ 7 & 2 \end{vmatrix} + 7\begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} = 3(2-3) - 4(4-21) + 7(2-7)$$

$$= 3(-1) - 4(-17) + 7(-5) = -3 + 68 - 35 = 68 - 38 = 30$$

Similarly, values of higher order determinants can be obtained. Expansion (or determination of value) of a determinant of higher order becomes increasingly complicated as the order of a determinant increases.

5.7 Properties of Determinants

We here merely state the properties of determinants. Students are advised to check them taking arbitrary numerical examples.

Property 1 : If the rows and columns of a determinant are interchanged, the value of the determinant will be unaffected.

Thus, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Property 2 : If all the elements in a row (or column) are zero, the value of the determinant is zero. That is, $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. Again, $\begin{vmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & 0 & c_3 \end{vmatrix} = 0$

Property 3 : If any two rows (or columns) are identical, the value of the determinant is zero. That is, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, Again, $\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$

More generally, if a row (or column) is a multiple of another row (or column), the value of the determinant is zero.

Property 4 : If any two rows (or columns) of a determinant are interchanged, the determinant changes its sign only and not the numerical value.

That is, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Again, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$

Property 5 : If all the elements in a row (or column) of a determinant are multiplied by the same number k, the value of the determinant is multiplied by k. Stated alternatively, if any row (or column) of a determinant has a common factor k to all its elements, then this common factor may be taken out and the value of the determinant will be k times the old one.

That is, $\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Again, $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Property 6 : If, to all the elements of a row (or column), we add a constant multiple of any other row (or column), the value of the determinant remains unaffected.

That is, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + ka_1 & c_1 \\ a_2 & b_2 + ka_2 & c_2 \\ a_3 & b_3 + ka_3 & c_3 \end{vmatrix}$

Again, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + ka_1 & b_2 + kb_1 & c_2 + kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Property 7 : If all the elements of a row (or column) of a determinant are expressed as the sum of two (or more) terms, the determinant can be expressed as the sum of two (or more) determinants.

That is, $\begin{vmatrix} a_1+k_1 & b_1 & c_1 \\ a_2+k_2 & b_2 & c_2 \\ a_3+k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$

5.8 Inverse of a Matrix

A square matrix A of order n with $|A| \neq 0$ is called a non-singular matrix. If $|A| = 0$, A is called a singular matrix. A non-singular matrix has a corresponding inverse matrix.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \text{ and } |A| \neq 0.$$

Then the inverse of the matrix A is given by, $A^{-1} = \frac{1}{|A|}$ Adj. A where Adj A = Transpose of the matrix of co-factors of A. We give an example.

Example 5.4 : Obtain inverse of matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Solution : Here $|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3 \times 2 - 1 \times 4 = 2$. Since $|A| \neq 0$, matrix A is non-singular and hence its inverse exists. $A^{-1} = \frac{1}{|A|}$. Adj. A where Adj A = Transpose of the matrix of cofactors of A.

Now, matrix of cofactors of $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

Transpose of the matrix of cofactors of A = Adj $A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$\therefore \ A^{-1} = \frac{1}{|A|}.\text{Adj } A = \frac{1}{2}\begin{matrix} 2 & 4 \\ 1 & 3 \end{matrix} = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

Two useful properties of inverse matrices are :

(i) $|A^{-1}| = \frac{1}{|A|}$. From our result, we see that $|A^{-1}| = \frac{1}{2} = \frac{1}{|A|}$.

(ii) If matrix A is symmetric, then its inverse is also symmetric. Further, $AA^{-1} = A^{-1}A = I$ (identity matrix). We may check it with our result.

$$\text{Check : } AA^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 3\times1+4\times-\frac{1}{2} & 3\times-2+4\times\frac{3}{2} \\ 1\times1+2\times-\frac{1}{2} & 1\times-2+2\times\frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{Again, } A^{-1}A = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\times3-2\times1 & 1\times4-2\times2 \\ -\frac{1}{2}\times3+\frac{3}{2}\times1 & -\frac{1}{2}\times4+\frac{3}{2}\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 4-4 \\ -\frac{3}{2}+\frac{3}{2} & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

5.9 Solution of Simultaneous Equations by Matrix Inversion Method

Suppose we have a system of n simultaneous linear equations as given below :

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = k_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = k_2$

....

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = k_n$

Here a_{ij}(i = 1, 2, ... m and j = 1, 2, ..., n) and k_i (i = 1, 2, ..., n) are constants. In matrix-vector form, the above system of equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ ... \\ k_n \end{bmatrix}$$

In matrix notation, $AX = B$ or, $X = A^{-1}B = \frac{1}{|A|}$ Adj. A.B

When B is a null vector, the system is called a homogeneous system. If B is non-null, the system is called non-homogeneous.

Now, consider the case when m = n. In that case, A is a square matrix. Also suppose that the system is non-homogeneous, i.e., B is non-null. We also assume that $|A| \neq 0$. Then, AX = B

or, $X = A^{-1}B = \frac{1}{|A|}$. Adj.A.B.

We have the previous result. The only difference is that in this case, m = n i.e., A is a square matrix. Our solution for X is known as Cramer's rule for solving linear equations. As an example, consider a system of two linear equations considered in section 5.7. We have,

$$a_1x + b_1y = k_1$$
$$a_2x + b_2y = k_2$$

In this case, we have seen that the solutions are as under :

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

We have noted that the determinant of the coefficients $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ comes as denominator in both the solutions. Let it be denoted by Δ. Thus, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of the system as it determines the solutions of the system. To get any meaningful value of x and y, the condition is, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. The solutions then by Cramer's rule be written as,

$$\frac{x}{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{1}{\Delta}$$

Using notations, we may write, $x = \dfrac{\Delta_1}{\Delta}$ and $y = \dfrac{\Delta_2}{\Delta}$ where $\Delta_1 = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$

and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Consider now the solution of the system by matrix inversion method. The system of equations is :

$a_1x + b_1y = k_1$
$a_2x + b_2y = k_2$

In matrix-vecor notations, it can be written as, $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

Using matrix-vector notations, we may write, AX = B

or, $X = A^{-1}B = \dfrac{1}{|A|}$.Adj.A.B.

We give an example.

Example 5. : Solve for x and y for the system,

$2x + 3y = 7$
$4x + 2y = 10$

Solution : By Cramer's rule, the solutions are as follows : here $a_1 = 2$, $b_1 = 3$, $a_2 = 4$, $b_2 = 2$, $k_1 = 7$, $k_2 = 10$

$$\text{Now, } x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 7 & 3 \\ 10 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}} = \frac{-16}{-8} = 2; \quad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 7 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}} = \frac{-8}{-8} = 1$$

So, x = 2, y = 1

Let us solve the system of equations by matrix inversion method. In matrix-vector notation, the system can be written as, $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$ i.e., AX = B

$\therefore\ X = A^{-1} B = \dfrac{1}{|A|}$. Adj A.B.

We have the coefficient matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ and $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = 4 - 12 = -8$

Now $A^{-1} = \dfrac{1}{|A|}$. Adj A $= \dfrac{1}{|A|}$. Transpose of cofactor matrix of A.

$$= \frac{1}{|A|}\begin{bmatrix} 2 & -4 \\ -3 & 2 \end{bmatrix}^T = -\frac{1}{8}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

$$\therefore\ X = A^{-1}B = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$\text{or, } X = \begin{bmatrix} -\frac{1}{4}\times 7 + \frac{3}{8}\times 10 \\ \frac{1}{2}\times 7 - \frac{1}{4}\times 10 \end{bmatrix} = \begin{bmatrix} \frac{15}{4} - \frac{7}{4} \\ \frac{7}{2} - \frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{8}{4} \\ \frac{2}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, we have, $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ i.e., x = 2, y = 1

Thus, we get the same result in Cramer's rule method and also in matrix inversion method.

We now give an example of a system of three linear equations involving three unknowns. Let us have the following system involving three unknons : x, y and z.

$a_{11}x + a_{12}y + a_{13}z = k_1$
$a_{21}x + a_{22}y + a_{23}z = k_2$
$a_{31}x + a_{32}y + a_{33}z = k_3$

The determinant of the coefficients of the unknowns is : $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The Cramer's rule method to solve a system of linear equations may be described as follows : First, the denominator in the solutions of the unknowns is the determinant of the coeficients of the unknowns. Second, the numerator in the solution for each unknown is the same as the determinant of the coefficients, with the exception that the column of coefficients of the particular unknown is replaced by the column of constants on the right hand side of the system of the equations.

Thus, the solution of the above system is : $\dfrac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{23} & a_{33} \end{vmatrix}}{\Delta}$, $y = \dfrac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{\Delta}$,

$z = \dfrac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{\Delta}$. Following this rule, we can solve a system of n simultaneous linear equations in n unknowns.

In the case of matrix inversion method, the procedure remains the same as before. In matrix-vector notation, the above system may be written as, $A_{3\times3}X_{3\times1} = B_{3\times1}$ so that $X = A^{-1}B = \dfrac{1}{|A|}$. Adj. A.B.

5.10 Jacobin Determinant, Hessian Determinant and Hessian Bordered Determinant

Suppose we want to optimise a bivariate function $Z = f(x_1, x_2)$. Also suppose that the two first order conditions $Z_1 = Z_2 = 0$ are met. Then, to optimise Z, two second order or sufficient conditions should also be met. They are as follows :

(i) $Z_{11} > 0$, $Z_{22} > 0$ for a minimum and $Z_{11} < 0$, $Z_{22} < 0$ for a maximum.

(ii) $Z_{11}.Z_{22} > (z_{12})^2$.

A convenient test for the second order condition is the Hessian matrix or, simply, the Hessian (named after the 19th century German mathematician Ludwig Otto Hesse). A Hessian |H| is a determinant composed of all the second order partial derivatives, with the second order direct partial derivatives on the principal diagonal and the second order cross partial derivatives off the principal diagonal. Thus, for our given bivariate function, $|H| = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}$ where $Z_{12} = Z_{21}$.

Now, if the first element on the principal diagonal, the first principal minor $|H_1| = Z_{11} > 0$ and if the second principal minor

$$|H_2| = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = Z_{11}Z_{22} - (Z_{12})^2 > 0,$$ the second order conditions for a minimum are met. When $|H_1| > 0$, $|H_2| > 0$, the Hessian is called positive definite. A positive definite Hessian fulfils the second order conditions for a minimum of an objective function.

On the other hand, if the first principal minor $|H_1| = Z_{11} < 0$ and the second principal minor

$$|H_2| = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} = Z_{11}Z_{22} - (Z_{12})^2 > 0,$$ the Hessian is negative definite. A negative definite Hessian fulfils the second order conditions for a maximum of the objective function.

Third Order Hessian : Suppose we have to optimise the multivariate function $Z = f(x_1, x_2, x_3)$. In this case the third order Hessian is, $|H| = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix}$. Then, if

$|H_1| = Z_{11} > 0$, $|H_2| = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} < 0$ and $|H_3| = |H| > 0$ where $|H_3|$ is the third principal minor, the Hessian |H| is positive definite. A positive definite Hessian fulfills the second order conditions for a minimum of the objective function. On the other hand, if $|H_1| < 0$, $|H_2| > 0$ and $|H_3| = |H| < 0$, the Hessian |H| is negative definite. A negative definite Hessian fulfils the second order conditions for a maximum of the objective function. In short, if the principal minors are all positive, |H| is positive definite and the second order conditions for a relative minimum are met. On the other hand, if the principal minors alternate sign between negative and positive (starting with negative sign), |H| is negative definite and the second order conditions for a relative maximum are met.

THE BORDERED HESSIAN FOR CONSTRAINED OPTIMISATION

Suppose we want to optimise $f(x_1, x_2)$ subject to the constraint $g(x_1, x_2)$. In that case, we form a Lagrange expression : $F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$ where λ is the Lagrange multiplier. The first order or necessary conditions to optimise $F(x_1, x_2, \lambda)$ require : $F_1 = F_2 = F_3 = 0$. We assume that the first order conditions are met. Then the second order conditions or sufficient conditions are to be met. The second order conditions can be expressed in terms of a bordered Hessian, $|\overline{H}|$, in either of the following two ways :

$$|\overline{H}| = \begin{vmatrix} F_{11} & F_{12} & g_1 \\ F_{21} & F_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} \text{ or } \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & F_{11} & F_{12} \\ g_2 & F_{21} & F_{22} \end{vmatrix}$$

The bordered Hessian determinant is simply the plain Hessian determinant $\begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix}$ boardered by the first derivatives of the constraint with zero on the principal diagonal. The order of a bordered principal minor is determined by the order of the principal minor which is being bordered. Hence $|\overline{H}|$ above represents a second order bordered principal minor $|\overline{H}_2|$, because the principal minor which is being bordered is (2 × 2).

Let us consider the second order conditions for optimisation of a multivariate function in n variables $f(x_1, x_2, ..., x_n)$ subject to the constraint $g(x_1, x_2, ..., x_n)$. In this case, the bordered Hessian, $|\overline{H}|$ can again be expressed as either of the two following ways :

$$|\overline{H}| \begin{vmatrix} F_{11} & F_{12} & ... & F_{1n} & g_1 \\ F_{21} & F_{22} & ... & F_{2n} & g_2 \\ ... & ... & ... & ... & ... \\ F_{n1} & F_{n2} & ... & F_{nn} & g_n \\ g_1 & g_2 & ... & g_n & 0 \end{vmatrix} \text{ or } \begin{vmatrix} 0 & g_1 & g_2 & ... & g_n \\ g_1 & F_{11} & F_{12} & ... & F_{1n} \\ g_2 & F_{21} & F_{22} & ... & F_{2n} \\ ... & ... & ... & ... & ... \\ g_n & F_{n1} & F_{n2} & ... & F_{nn} \end{vmatrix}$$

where $|\overline{H}| = |\overline{H}_n|$, because of the (n × n) principal minor being bordered.

Now, if $|\overline{H}_2|, |\overline{H}_3|, ..., |\overline{H}_n| < 0$, the bordered Hessian is positive definite, which is a sufficient condition for a minimum. It should be noted that the test starts with $|\overline{H}_2|$, and not $|\overline{H}_1|$.

On the other hand, if $|\overline{H}_2| > 0$, $|\overline{H}_3| < 0$, $|\overline{H}_4| > 0$, etc., the bordered Hessian is negative definite, which is a sufficient condition for a maximum. Thus, if a given Hessian bordered determinant $|\overline{H}|$ meets the above mentioned criteria, we are assured of a minimum or a maximum of our objective function. But if those criteria are not met, further tests are required, since the given criteria represent sufficient conditions, and not necessary conditions.

5.11 Applications of Matrix and Determinant Operations in Economics

Matrix and determinant operations have so many applications in Economics. We shall here consider their applications in the context of derivation of Slutsky equation, Leontief static open model and solving IS-LM model. Let us consider them one by one.

5.11.1 Derivation of Slutsky Equation

We know that if price of a commodity falls, *ceteris paribus*, its quantity demanded rises, and *vice versa*. This is known as price effect. Indifference curve theorists like Hicks, Slutsky, Allen et. al. argue that this price effect can be decomposed into an income effect and a substitution effect. As the price of a commodity falls, it becomes relatively cheap than its substitutes. So, the consumer purchases more of the commodity, giving up some amounts of its substitutes. This is substitution effect which measures the effect of change in relative price of a commodity, real income remaining the same. On the other hand, as the price of a commodity falls, real income or purchasing power of the consumer rises. Then also demand for the commodity changes. This is income effect which measures the effect of change in real income of the consumer, relative price of the commodities remaining the same. Price effect is the sum of income effect and substitution effect (price effect = substitution effect + income effect). The Slutsky equation (named after Russian mathematician Eugene Slutsky) shows the relationship among the price effect, income effect and substitution effect mathematically. The equation states that price effect = substitution effect + income effect. Mathematically,

$$\left(\frac{\partial q_1}{\partial p_1}\right) = \left(\frac{\partial q_1}{\partial p_1}\right)_{\text{U constant}} - q_1\left(\frac{\partial q_1}{\partial y}\right)_{\text{prices constant}}$$

Let us try to deduce this equation. Suppose the consumer wants to maximise utility (U) by consuming two goods, Q_1 and Q_2. Their respective quantities are q_1 and q_2 and prices are p_1 and p_2. Let the given money income of the consumer be y. Thus, formally our problem is to maximise $U = f(q_1, q_2)$ subject to the income constraint or budget constraint : $y = p_1q_1 + p_2q_2$. So it is a problem of constrained maximisation. We form the following Lagrange expression :

$V = f(q_1, q_2) + \lambda(y - p_1q_2 - p_2q_2)$ where λ = Lagrange multiplier.

The first order conditions to maximise V require,

$$\frac{\partial V}{\partial q_1} = 0 \text{, or, } f_1 - \lambda p_1 = 0 \qquad ...(1)$$

$$\frac{\partial V}{\partial q_2} = 0 \text{ , or, } f_2 - \lambda p_2 = 0 \qquad ...(2)$$

$$\frac{\partial V}{\partial \lambda} = 0\text{, or, } y - p_1q_1 - p_2q_2 = 0 \qquad ...(3)$$

Taking total derivative of them and re-arranging, we get,

$$f_1dq_1 + f_{12}dq_2 - p_1d\lambda = \lambda dp_1 \qquad ...(4)$$

$$f_{21}dq_1 + f_{22}dq_2 - p_2d\lambda = \lambda dp_2 \qquad ...(5)$$

$$-p_1dq_1 - p_2dq_2 = -dy + q_1dp_1 + q_2dp_2 \qquad ...(6)$$

In matrix-vector form, the above system of equations can be written as,

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1dp_1 + q_2dp_2 \end{bmatrix}$$

We can solve these three equations for dq_1, dq_2 and $d\lambda$ by Cramer's rule. Then the terms on the R.H.S must be treated as constants. Let D represent the determinant of the coefficient matrix. Let D_{ij} represent the cofactor of the element in the ith row and jth column. Then

$$dq_1 = \frac{\lambda dp_1D_{11} + \lambda dp_2D_{21} + (-dy + q_1dp_1 + q_2dp_2)D_{31}}{D} \qquad ...(7)$$

$$dq_2 = \frac{\lambda dp_1D_{12} + \lambda dp_2D_{22} + (-dy + q_1dp_1 + q_2dp_2)D_{32}}{D} \qquad ...(8)$$

We now consider equation (7). We suppose that p_1 and p_2 do not change and only y (income) changes. Then, $dp_1 = dp_2 = 0$.

$$\text{So, } \partial q_1 = \frac{-\partial y.D_{31}}{D} : \left(\frac{\partial q_1}{\partial y}\right)_{\substack{\text{prices}\\\text{constant}}} = -\frac{D_{31}}{D} \qquad ...(9)$$

This equation shows the effect of change in income on the quantity demanded of Q_1, prices remaining the same. This gives us the income effect.

We now consider the substitution effect. In the Hicksian measure of substitution effect, total utility of the consumer remains the same, i.e., $dU = 0$.

Now, we have the utility function : $U = f(q_1, q_2)$

So, $dU = f_1dq_1 + f_2dq_2$

Putting $dU = 0$, we get, $-\dfrac{dq_2}{dq_1} = \dfrac{f_1}{f_2}$ or, MRS = $\dfrac{f_1}{f_2}$.

Again, from (1) and (2), $\frac{f_1}{f_2} = \frac{p_1}{p_2}$.

Thus, in equilibrium, $-\frac{dq_2}{dq_1} = \frac{f_1}{f_2} = \frac{p_1}{p_2}$

So, $p_1dq_1 + p_2dq_2 = 0$

Hence, from (6), $- dy + q_1dp_1 + q_2dp_2 = 0$

Again, as p_2 is unchanged, $dp_2 = 0$. Then from (7), we get,

$$q_1 \quad \frac{p_1}{D}.D_{11}\text{, or, } \left(\frac{\partial q_1}{\partial p_1}\right)_{U\,constant} = \frac{\lambda D_{11}}{D} \qquad ...(10)$$

This equation gives us the substitution effect of the fall in price of Q_1

On the other hand, if price of Q_1 only changes while price of Q_2 and income remaining the same, we get the price effect. In this case, $dp_2 = 0$ and $dy = 0$.

Then from (7), we get, $\partial q_1 = \frac{\lambda D_{11}\delta p_1 + q_1\delta p_1 D_{31}}{D}$

$$\text{or, } \frac{\partial q_1}{\delta p_1} = \frac{\lambda D_{11}}{D} + \frac{q_1 D_{31}}{D} \qquad ...(11)$$

Now combining (9), (10) and (11) we get,

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial p_1}\right)_{U=constant} - q_1\left(\frac{\delta q_1}{\partial y}\right)_{prices=constant} \qquad ...(12)$$

This equation is known as Slutsky equation. The L.H.S. of this euqation represents price effect. The first term of the R.H.S. represents the substitution effect while the second term represents the income effect. Thus, the Slutsky equation shows that the price effect is the resultant of the substitution effect and the income effect.

We now consider the direction of these effects. We know that our second order condition of utility maximisation requires that $D > 0$. Further, Lagrange multiplier, $\lambda = MU_m > 0$. Now, the substitution effect = $\frac{\lambda D_{11}}{D}$.

But $D_{11} = \begin{vmatrix} f_{22} & -p_2 \\ -p_2 & 0 \end{vmatrix} = -p_2^2 < 0$. Thus, the substitution effect as always is negative as λ

> 0, D > 0. The income effect $= -q_1\left(\frac{\partial q_1}{\partial y}\right)_{\substack{\text{prices}\\ \text{constant}}} = q_1\frac{D_{31}}{D}$.

Now, $D_{31} = \begin{vmatrix} f_{12} & -p_1 \\ f_{22} & -p_2 \end{vmatrix} = (-p_2 f_{12} + p_1 f_{22}) \gtrless 0.$

Thus the sign of income effect is indetermenate. Income effect with respect to price is negative for a normal good but positive for an interior good. Thus, if Q_1 is a normal good, both the substitution effect and the income effect in equation (12) have negative signs so that, $\frac{\partial q_1}{\partial p_1} < 0$, and the demand curve is downward sloping. But in the case of inferior goods, the first term on the R.H.S. of equation (12) is negative while the second term is positive. Hence, $\frac{\partial q_1}{\partial p_1}$ may be positive or negative. If the second term is positive and stronger than the first term, $\frac{\partial q_1}{\partial p_1} > 0$ and the demand curve is upward rising. This is the case of Giffen goods where the income effect is negative and stronger than the substitution effect.

In the similar manner, we can analyse the effect of change in p_2 on Q_2 from equation (8).

5.11.2 Leontief Static Open Model

Wasilly Leontief has done an input-output analysis which shows interdependence among different sectors of an economy. We know that output of one sector goes to the other sector as input. Hence, there should be a balance of demand and supply made by different industries. Leontief has shown this in terms of a model. Here we shall consider Leontief static open model (LSOM). A static model is concerned with the determination of output at a particular period of time. Again, in an open model, some of the relevant variables are exogeneous while others are endogeneous. Our simplified model is based on the following assumptions :

(1) There are two industries or sectors in the economy. The product of one industry is used as input in the production of other, i.e., there is interdependence between them.

(2) Total demand for each product is equal to its gross output or supply.

(3) Total demand for each product has to components : intermediate demand and final demand.

(4) There is only one factor of production, labour.

(5) Input co-efficients are technologically fixed.

(6) Each sector produces only one commodity and there is no joint product.

In the model, we use the following notations :

x_j = output in the jth sector (j = 1, 2)

x_{ij} = part of output of the i-th secotr used as input in the j-th sector (i, j = 1, 2).

C_j = consumption demand or final demand for the j-th product (j = 1, 2).

L = supply of labour (only primary factor)

$a_{ij} = \frac{x_{ij}}{x_j}$ = input coefficient representing the amount of i-th commodity required as input for unit production of the j-th commodity.

$a_{oj} = \frac{L_j}{X_j}$ = labour coefficient representing the amount of labour required to produce one unit of the j-th commodity.

Let us discuss the Leontief open static model (LSOM). Using supply-demand equality for sector 1, we get,

$X_1 = X_{11} + X_{12} + C_1$, or $X_1 = a_{11}X_1 + a_{12}X_2 + C_1$ ...(1)

Similarly, for sector 2,

$X_2 = X_{21} + X_{22} + C_2$, or $X_2 = a_{21}X_1 + a_{22}X_2 + C_2$ ...(2)

For equilibrium in the labour market, supply of labour = demand for labour,

i.e., $L = L_1 + L_2$

or, $L = a_{01}L_1 + a_{02}L_2$ ...(3)

We may write equation (1) and (2) in matrix-vector form,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Using matrix-vector notations : X = AX + C.

or, (I–A)X = C where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or, $X = (I - A)^{-1}.C$, or, $X = \frac{1}{|I - A|}.\text{Adj.}(I - A).C$

So, to get X, we have to find out $(I-A)^{-1}$.

We first determine $|I-A|$.

$$|I-A| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}$$

$\therefore$ $|I-A| = (1-a_{11})(1-a_{22}) - a_{21}a_{12} = \Delta$(say). We assume $\Delta \neq 0$.

Now, Adj $(I-A)$ = Transpose of matrix of cofactors of $(I-A)$

$$\text{Matrix of cofactors of } (I-A) = \begin{bmatrix} 1-a_{22} & a_{21} \\ a_{12} & 1-a_{11} \end{bmatrix}$$

$$\therefore \text{Adj}(I-A) \quad \begin{matrix} 1 & a_{22} & a_{21} \\ a_{12} & 1 & a_{11} \end{matrix}^{T} = \begin{bmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{bmatrix}$$

$$\text{Now, } X = \frac{1}{|I-A|}.\text{Adj}(I-A).C$$

$$\text{or, } \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 1-a_{22} & a_{12} \\ a_{21} & 1-a_{11} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\therefore X_1 = \frac{1}{\Delta}[(1-a_{22})C_1 + a_{12}C_2], \text{ and } X_2 = \frac{1}{\Delta}[a_{21}C_1 + (1-a_{11})C_2]$$

Total demand for labour = $a_{01}X_1 + a_{02}X_2 = L$ = Labour supply.

HAWKINS-SIMON CONDITION

In Leontief open static model, we have, $X = (I-A)^{-1}C$

$$\text{or, } X = \frac{1}{|I-A|}.\text{Adj}.(I-A).C.$$

So, to have positive output, $|I-A| = \Delta$(say) must be positive,

i.e., $\Delta = (1-a_{11})(1-a_{22}) - a_{21}a_{12} > 0$ and $(1-a_{11}) > 0$, $(1-a_{22}) > 0$.

All these are known as Hawkins-Simon condition. Let us see the economic implication of these conditions. One condition is : $1-a_{11} > 0$, or, $a_{11} < 1$. It implies that the amount of first commodity required to produce one unit of the first commodity should be less than one. Otherwise there will be no justification of production. Same interpretation may be given for $(1-a_{22}) > 0$. Let us consider the third contion : $(1-a_{11})(1-a_{22}) > a_{22}a_{12}$ or, $(1-a_{11}) > a_{21}a_{12}$ or, $a_{11} + a_{21}a_{12} < 1$. Now, $a_{21}.a_{12}$ is the indirect requirement of the first commodity for unit production of the first commodity. The condition

$(a_{11} + a_{21}a_{12}) < 1$ then states that the direct (a_{11}) and indirect requirements of the first commodity for unit production of the first commodity should be less than one unit of that commodity. Otherwise there is no logic or justification of production.

Example 5.5 : An economy uses coal and steel to produce coal and steel. Suppose, 0.4 tonne of steel 0.7 tonne of coal are required to produce on tonne of steel. Similarly, 0.1 tonne of steel and 0.6 tonne of coal are required to produce one tonne of coal. Is the system viable?

Again, 2 and 5 labour days are needed to produce one unit of coal and steel respectively. If the economy requires 100 tonnes of coal and 50 tonnes of steel for consumption, calculate gross output and required labour.

Solution : We denote steel industry as sector 1 and coal industry as sector 2 and summarise the given infomation below. We also denote output levels of steel and coal as X_1 and X_2 respectively and their final consumptions as C_1 and C_2 respectively.

	Steel	Coal	Final demand
Steel	$0.4(a_{11})$	$0.1(a_{12})$	$50(C_1)$
Coal	$0.7(a_{21})$	$0.6(a_{22})$	$100(C_2)$
Labour	$5(a_{01})$	$2(a_{02})$	–

Applying the equality condition between supply and demand, we get,

$X_1 = a_{11}X_1 + a_{12}X_2 + C_1$ ...(1)

$X_2 = a_{21}X_1 + a_{22}X_2 + C_2$ ...(2)

$L = a_{01}X_1 + a_{02}X_2$ ...(3)

Equations (1) and (2) can be written in matrix form as :

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \text{ or } \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

or $X = AX + C$ or, $(I - A)X = C$

$\therefore X = (I - A)^{-1}C$

or, $X = \dfrac{1}{|I - A|} \text{Adj. } (I - A).C$

Here, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

$$\therefore\ (I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$$

Further, $|I - A| = \begin{vmatrix} 0.6 & 0.1 \\ 0.7 & 0.4 \end{vmatrix} = 0.24 - 0.07 = 0.17$

Here (i) the diagonal elements of |I – A| are all positive.

(ii) The determinant |I – A| is positive. So the system is viable.

Let us solve the system for X_1, X_2 and L. To do that, we have to determine $(I - A)^{-1}$. For that, we first determine cofactor matrix of (I – A)

The cofactor matrix of $(I - A) = \begin{bmatrix} 0.4 & 0.7 \\ 0.1 & 0.6 \end{bmatrix}$

Now, Adj.(I – A) = transpose of cofactor matrix of $(I - A) = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

Now, $X = (I - A)^{-1}C$ or, $X = \dfrac{1}{|I - A|}.\text{Adj}\,(I - A).C$

or, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \dfrac{1}{0.17}\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 50 \\ 100 \end{bmatrix}$

$X_1 = \dfrac{1}{0.17}(0.4 \times 50 + 0.1 \times 100) = \dfrac{30}{0.17} = 176.5$

$X_2 = \dfrac{1}{0.17}(0.7 \times 50 + 0.6 \times 100) = \dfrac{95}{0.17} = 558.8$

Total demand for labour = $L = a_{01}X_1 + a_{02}X_2 = 5 \times 176.5 + 2 \times 558.8 = 2{,}000$ (approx)

5.11.3 Cramer's Rule for Solving Problems in IS-LM Model

In the IS-LM model, the equilibrium rate of interest (r) and the equilibrium level of income(Y) are simultaneously determined by intersection point between IS and LM curves. The IS curve is the investment-saving (I-S) equality curve. In the simple case, we assume that I = I(r) such that $I'(r) < 0$ and S = S(Y) such that $0 < S'(Y) < 1$. So the equation of the IS curve is : I(r) = S(Y), or, I(r) – S(Y) = 0. On the other hand, the LM curve is the curve representing the equality between money demand (L) and money supply (M). Money demand has two components : demand for active balance or transaction-precautionary demand for money

(L_1) and demand for idle balance or speculative demand for money (L_2). It is assumed that $L_1 = L_1(Y)$ such that $L_1'(Y) > 0$ and $L_2 = L_2(r)$ such that $L_2'(r) < 0$. Money supply is assumed to be autonomously given at M_0. So, the equation of the LM curve is : $L_2 + L_1 = M_0$, or $L_2(r) + L_1(Y) = M_0$. Thus, we get a system of two simultaneous equations involving two unknowns : r and Y. The equations are :

$$I(r) - S(Y) = 0$$

$$L_2(r) + L_1(Y) = M_0$$

We assume that all the functions involved in the system are linear. Now, applying Cramer's rule, we can easily solve the system for r and Y and thus determine the equilibrium rate of interest and the equilibrium level of income.

Let us give an algebraic example.

Suppose, our investment function is : $I(r) = \alpha - ir$ and the saving function is : $S(Y) = -a + sY$. Putting $I(r) = S(Y)$, we get, $\alpha - ir = -a + sY$ $[(\alpha, i, a, s) > 0]$

or, $ir + sy = a + \alpha = A$ where $A = a + \alpha$ = total autonomous expenditure. This is our IS curve.

Let us consider the equation of the LM curve.

Let the demand for active balance be, $L_1 = l_1Y$ and the demand for idle balance be, $L_2 = M_1 - l_2r$. Now, putting $L_2 + L_1 = M_0$ (given money supply), we get,

$M_1 - l_2r + l_1Y = M_0$ (Here $l_1, l_2, M_1, M_0 > 0$ and $M_0 > M_1$)

or, $l_2r - l_1Y = M_1 - M_0 = -(M_0 - M_1) = -M$(say) where $M_0 - M_1$ is denoted by M.

Thus, we have two simultaneous linear equations in two unknowns. They are :

$$ir + sY = A$$

$$l_2r - l_1Y = -M$$

Now applying Cramer's rule, we can solve them for r and Y.

$$r = \frac{\begin{vmatrix} A & s \\ -M & -l_1 \end{vmatrix}}{\begin{vmatrix} i & s \\ l_2 & -l_1 \end{vmatrix}} = \frac{-Al_1 + sM}{-il_1 - sl_2} \quad \text{and} \quad Y = \frac{\begin{vmatrix} i & A \\ l_2 & -M \end{vmatrix}}{\begin{vmatrix} i & s \\ l_2 & -l_1 \end{vmatrix}} = \frac{-iM + Al_2}{-il_1 - sl_2}$$

Alternatively, we may solve the system by matrix-inversion method. In matrix-vector form, the system can be written as : $\begin{bmatrix} i & s \\ l_2 & -l_1 \end{bmatrix}\begin{bmatrix} r \\ Y \end{bmatrix} = \begin{bmatrix} A \\ -M \end{bmatrix}$

Using symbols, $BX = C \;\therefore\; X = B^{-1}.C$

Let us solve the system by Cramer's rule. We give a numerical example.

Example 5.6 : Suppose we have,

$S = -90 + 0.375Y$

$I = 150 - 100r$

$L_1 = 0.25Y,$

$L_2 = 50 - 200r$

$M_0 = 180$

Determine the equilibrium rate of interest and the equilibrium level of income.

Solution : Putting I = S, we get, $150 - 100r = -90 + 0.375Y$

or, $100r + 0.375Y = 240$. This is our IS curve.

Let us deduce the LM curve. On the LM curve, L = M

or, $L_1 + L_2 = M$

So, $0.25Y + 50 - 200r = 180$

or, $200r - 0.25Y = -130$. This is our LM curve.

Thus, we have,

$100r + 0.375Y = 240$ ... (IS curve)

$200r - 0.25\,Y = -130$ ... (LM curve)

By Cramer's rule, $r = \dfrac{\begin{vmatrix} 240 & 0.375 \\ -130 & -0.25 \end{vmatrix}}{\begin{vmatrix} 100 & 0.375 \\ -200 & -0.25 \end{vmatrix}} = \dfrac{240 \times -0.25 + 130 \times 0.375}{100 \times -0.25 - 200 \times 0.375} = \dfrac{60 \quad 48.75}{25 \quad 75}$

$= \dfrac{-11.25}{-100} = 0.1125 = 11.25\%$

$Y = \dfrac{\begin{vmatrix} 100 & 240 \\ 200 & -130 \end{vmatrix}}{\begin{vmatrix} 100 & 0.375 \\ 200 & -0.25 \end{vmatrix}} = \dfrac{100 \times -130 - 200 \times 240}{-25 - 75} = \dfrac{13{,}000 \quad 48{,}000}{100} = \dfrac{61{,}000}{100} = 610$

5.12 Summary

1. Definition and Concept of a Matrix

Any rectangular array of numbers is called a matrix. A matrix with a single row is called a row matrix or a row vector. Similarly, a matrix with single column is called a column matrix or a column vector. There are some specific rules for the operations of matrices, i.e., for addition, subtraction, multiplication, etc.

2. Different Types of Matrices

Some of the major types of matrices are : column matrix, row matrix, transposed matrix, square matrix, symmetric matrix, diagonal matrix, identity or unit matrix, orthogonal matrix, idempotent matrix, etc.

3. Determinant of a Matrix and its Associated Concepts

A determinant is simply described as a square array of numbers. It is so called as it is used in the determination of the solution of a system of simultaneous equations. To every square matrix, there corresponds a number known as the determinant of that matrix. Some associated concepts used to make operations with determinants are : principal diagonal, minor, cofactor, etc. There are some properties of determinants which are very helpful especially for evaluating determinants.

4. Inverse of a Matrix

A square matrix A with $|A| \neq 0$ is called a non-singular matrix. A non-singular matrix has a corresponding inverse matrix. The concept of inverse matrix is useful to solve a system of linear simultaneous equations. An alternative method of solving a system of linear equations is Cramer's rule method.

5. Applications of Matrix and Determinant Operations in Economics

In Economics, there are numerous applications of matrices and determinants. In fact, in any economic or econometric model, whenever we use some simultaneous equations and/or some notations, use of matrices and determinants is very much helpful to deal with them. In the present unit, we have considered three applications of matrix and determinant operations, namely, derivation of Slutsky equation, Leontief static open model (LSOM) and solution of IS-LM model.

5.13 Excercises

Short Answer Type Questions

1. Define matrix.
2. What is column matrix and what is row matrix?
3. What is column vector and what is row vector?
4. What is a square matrix?
5. What is a symmetric matrix?
6. Define a diagonal matrix.
7. What is an identity matrix?
8. Define a unit matrix.
9. What is null matrix?
10. What is an idempotent matrix?
11. What is a determinant?
12. What is principal diagonal of a determinant?
13. What is minor in the context of a determinant?
14. What is cofactor in relation to a determinant?
15. In the context of Leontief input-output analysis, what is a static model?
16. Why is Leontief static open model called 'open'?

Medium Answer Type Questions

1. Explain the concept of a matrix.
2. Show the addition operation of a matrix, taking a simple example.
3. What is transposed matrix? Give example.
4. Prove that $(AB)' = B'A'$
5. Illustrate the concept of a square matrix.
6. Explain the concept of symmetric matrix.
7. Illustrate the concept of diagonal matrix.
8. Briefly explain the concept of identity matrix.
9. Explain the concepts of minor and cofactor of a determinant.
10. Briefly discuss the implications of Hawking-Simon condition in the context of Leontief static open model (LSOM).

11. Distinguish between a matrix and a determinant.

12. Find Adj. A of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

13. Taking any two matrices of order (2×2), show that $AB \neq BA$.

14. If $A = \begin{bmatrix} 2 & -3 \\ 4 & -11 \end{bmatrix}$, find A^{-1}.

15. $2x + 5y = 24$ and $3x + 8y = 38$. Solve by matrix inversion method.

Long Answer Type Questions

1. Explain with a suitable example the multiplication of two matrices.
2. Show with a suitable example that transpose of transpose of a matrix is the original matrix. Also prove that $(A \pm B)' = A' \pm B'$.
3. State the major properties of a determinant.
4. Write a short note on expansion of a determinant.
5. Explain how can you find out the inverse of a matrix. Show how the matrix inversion method can be used to solve a system of simultaneous equations.
6. Briefly explain the Cramer's rule method to solve a system of simultaneous equations involving two variables.
7. Write a short note on Hessian determinant and Bordered Hessian determinant.
8. What is Slutsky equation? Derive the equation and interprete its various terms.
9. Briefly discuss the Leontief static open model stating clearly its assumptions.
10. Explain how Cramer's rule may be used to solve for the variables in an IS-LM model.
11. Solve the following system by matrix inversion method :

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

12. Solve the following system by Cramer's rule method :

$$2x - y + 2z = 6$$
$$x - 2y + 3z = 6$$
$$3x - 3y - z = -6$$

13. Solve the matrix equation AX = B where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

14. Calculate output in the two sectors on the basis of following data to meet final demand of 200 and 800 units of Agriculture and industry, respectively.

Sectors	Purchasing sector		Current demand
	Agriculture	Industry	
Agriculture	300	600	100
Industry	400	1,200	400

15. Find total output for each industry if the new final demands are 180 and 440 units respectively

Industry	Input to		Final demand
	Industry 1	Industry 2	
1	160	200	40
2	80	400	320

16. Our equations in commodity and money markets, respectively are :

$$200r + 0.36Y = 380$$

$$200r - 0.40Y = -380$$

Determine equilibrium r and Y both by matrix inversion method and Cramer's rule method.

17. The IS-LM model is :

$S(Y) = I(r) + G$

$L_1(Y) + L_2(r) = M$

Find $\frac{dY}{dG}$, $\frac{dr}{dG}$, $\frac{dY}{dM}$ and $\frac{dr}{dM}$.

18. Solve the national income model :

$Y = C + I_0 + G_0$ and $C = a + bY$

using Cramer's rule and also by matrix-inversion method.

19. Consider the following model.

$C = a + bY$

$I = d + eY$

$Y = C + I$

Solve for the endogeneous variables using matrix form and also using Cramer's rule.

20. C = 0.75Y + 2,000, I = 0.15Y + 3,000

Determine the equilibrium level of income both by Cramer's rule and by matrix inversion method.

5.14 References

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Unit 6 ❒ Dynamic Analysis

Structure

6.1 Objectives

After studying the unit, the reader will be able to know

- First and second order difference equations and their solutions
- Application of difference equations in Economics
- Solution of a first order differential equation
- Application of differential equation in Economics

5.2 Introduction

There are two basic approaches to examine the course of a system of economic variables through time. One is the static analysis and the other is the dynamic analysis. In the static analysis, all the variables involved refer to the same point of time or the same period of time. A variant of the static analysis is known as comparative static analysis in which we compare the values of relevant variables in two or more static situations. Thus, comparative static analysis is also basically a static analysis. On the other hand, if the economic variables involved refer to different points of time or, different periods of time, then the analysis is called dynamic analysis. In static analysis, all the variables involved refer to the same period of time. Further, the time element is not considered in the process of determining the equilibrium values of the variables. Static analysis considers the determination of an equilibrium position. It is not concerned with the time required to achieve that equilibrium position. It does not also consider the path by which the variables approach their equilibrium values. All these are considered in dynamic analysis. In this analysis, all the variables are dated. Hence we can know the time path of an economic variable in this case.

For example, suppose we assume that demand for any commodity in any period of time is a function of current price while its supply depends on the price of the previous period i.e., $D_t = D(p_t)$ while $S_t = S(p_{t-1})$ and it represents a dynamic relationship since it shows a relation between prices in two successive periods of time. Such an equation is called a difference equation. Solving it we can get the time path of price (p). The time path of p represents the path along which price movement will take place over a period of time. Thus, from the dynamic analysis, we can know the pattern of movement of an economic variable from one equilibrium situation to another.

In particular, dynamic analysis is necessary for three reasons. **First**, adjustment of one variable to bring change in other takes time. Hence there are lags in many functions. The presence of these lags necessitates the use of dynamic analysis. **Secondly**, there are certain variables which depend, among other things, on the rate of change of some other variables. For example, demand for any commodity may depend on the rate of change of price of that commodity. Such problems involving rates of growth requires dynamic analysis. **Thirdly**, dynamic analysis is also necessary for considering the stability of equilibrium. An equilibrium is said to be stable if, after some disturbance or change, it reaches to an equilibrium position. Whether the system moves towards equilibrium or not, depends on the time path of relevant variable. And this time path can only be determined from dynamic analysis.

6.3 Use of Difference Equation

We have mentioned that in dynamic analysis, the variables involved refer to different points of time, or different periods of time. A dynamic model is concerned with the change in relevant variables over time. This model can be formulated in two alternative ways : in period terms or in continuous terms. In period analysis, the flow of time is divided into successive discrete periods of finite constant length taken as units of time. For example, a variable price is written as p_t for periods t = 0, 1, 2, 3, etc. In this case, various relations and conditions of a dynamic model are expressed in terms of difference equations. On the other hand, in continuous analysis, time flows continuously in an endless manner. Each variable is then taken as a continuous and differentiable function of time. Naturally, in continuous analysis, a dynamic model uses differntial equations, rather than a difference equation. The choice of the specific analysis is mainly a matter of mathematical convenience. We shall consider the solution of difference equation which takes time as a discrete variable. We shall first illustrate the solution of a first order diffence equation, and then the solution of a second order difference equation. After that, we shall consider the solution of a differential equation which considers time as a continuous variable.

6.3.1 Solution of a First Order Difference Equation

The general form of a linear non-homogeneous difference equation of n-th order is :

$$a_0Y_t + a_1Y_{t-1} + a_2Y_{t-2} + ... + a_nY_{t-n} + C= 0$$

If C = 0, it is called linear homogeneous difference equation of order n. The form of a first order linear non-homogeneous difference equation is :

$$a_0Y_t + a_1Y_{t-1} + C = 0$$

We shall consider its solution. We shall first follow the general method and then a relatively rudimentary method, called iterative method. The solution of such an equation has two parts : general solution = homogeneous solution and particular solution , i.e., $Y_t = Y_c + Y_p$.

Let the simple form of the difference equation be,

$$Y_{t+1} + aY_t = C$$

For homogeneous solution, we consider the homogeneous part i.e., we take, C = 0.

The homogeneous solution is also called complementary solution (Y_c). Then,

$Y_{t+1} + aY_t = 0$.

Let $Y_1 = Hb^t$ be the complementary solution. Then putting

$Y_t = Hb^t$, we get, $Hb^{t+1} + aHb^t = 0$

or, $b^{t+1} + ab^t = 0$, $(H \neq 0)$. Then $b + a = 0$, $(b^t \neq 0)$

Then, $b = -a$

So, the complementary solution is : $Y_t = Hb^t = H(-a)^t$.

We now consider the particular solution, Y_p.

Let $Y_t = K$(a constant) be the particular solution. As K is a constant, Y = K will hold for all t.

$$\therefore\ Y_{t+1} + aY_t = C \ \therefore\ K + aK = C \text{ or, } K = \frac{C}{1+a}$$

Then, our particular solution is : $Y_p = \dfrac{C}{1+a}$, $(a \neq -1)$.

Now, the general solution is :

$$Y_t = Y_c + Y_p = H(-a)^t + \frac{C}{1+a}, \ (a \neq -1)$$

The value of H is to be determined from the initial condition, i.e., by putting t = 0. Then we have, $Y_0 = H + \dfrac{C}{1+a} \ \therefore\ H = Y_0 - \dfrac{C}{1+a}$

Thus our final solution is, $Y_t = H(-a)^t + \dfrac{C}{1+a}$

$$\text{or, } Y_t = \left[Y_0 - \frac{C}{1+a}\right](-a)^t + \frac{C}{1+a}, \ (a \neq -1)$$

If $a = -1$, then we shall try the particular solution, $Y_t = Kt$, instead of $Y_t = K$.

Then $Y_{t+1} + aY_t = C$

$\therefore K(t + 1) + aKt = C$

$\therefore Kt + K + aKt = C$

or, $K = C$ as $a = -1$

So, $Y_p = Ct$ is our particular solution.

Then our final solution is : $Y_t = Y_C + Y_p$

or, $Y_t = H(-a)^t + C.t$ where the value of H is to be determined from the initial condition, i.e., putting t = 0. Then, $Y_0 = H$

So, $Y_t = Y_0 + Ct$

This is our total solution in the case of a = –1.

Example 6.1 : Solve $Y_t = 2Y_{t-1} + 3$

Solution : This is a first order non-homogeneous difference equation. Its total solution is :

$Y_t = Y_C + Y_P$.

For solution of the homogeneous part, we take, $Y_t = 2Y_{t-1}$.

Let $Y_t = Hb^t$ be a solution. Then $Hb^t = 2Hb^{t-1}$

$\therefore$ b = 2. So, $Y_C = Hb^t = H2^t$

Now we consider the particular solution.

Let $Y_t = K$ be the particular solution. Then it will hold for all t, i.e., $Y_t = Y_{t-1} = K$.

Now we have, $Y_t = 2Y_{t-1} + 3$

$\therefore$ K = 2K + 3 or, K = –3

So, $Y_p = -3$ is the particular solution.

Hence, general solution, $Y_t = H2^t - 3$ where the value of H is to be determined from the initial condition.

Putting t = 0, $Y_0 = H - 3$ $\therefore$ $H = Y_0 + 3$

So the final solution is : $Y_t = [Y_0 + 3]2^t - 3$

Example 6.2 : Solve $Y_{t+1} = Y_t + 1$ when $Y_0 = 10$

Solution : We first consider the solution of the homogeneous part, i.e., $Y_{t+1} = Y_t$

Let $Y_t = Hb^{t+1} = Hb^t$

$\therefore$ b = 1, (H $\neq$ 0)

So, $Y_t = H(1)^t = H$.

Thus, $Y_C = H$ is the complementary solution.

Now, we consider the particular solution, Y_p.

Here Y_t = K(a constant) cannot be a solution.

For, then $Y_{t+1} = Y_t = K$. So, K = K + 1 i.e., 1 = 0 which is absurd. So we try another solution.

Let $Y_t = K.t$ be a solution,

Then, K(t + 1) = Kt + 1 $\therefore$ K = 1

Thus, our particular solution is : $Y_p = K.t = t$

So, general solution is : $Y_t = Y_C + Y_p = H + t$

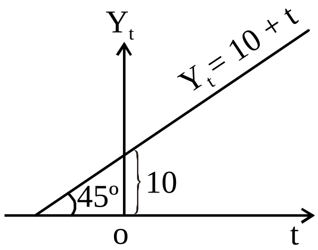
where the value of H is to be determined from the initial condition (i.e., by putting t = 0)

$\therefore$ $Y_0 = H$

So, our solution is : $Y_t = Y_0 + t$. But it is given that $Y_0 = 10$

$\therefore Y_t = 10 + t$ is our solution.

In our figure 6.1, we have drawn the time path of Y. It is an upward rising straight line with a slope equal to 1(= tan 45°) and a positive vertical intercept equal to 10 when plotted against t.



(Fig. 6.1)

Example 6.3. : Given $Y_{t+1} = \alpha Y_t - \beta$. Find the time path of Y.

Solution : We first consider the complementary solution : $Y_{t+1} = \alpha Y_t$

Let $Y_t = Hb^t$ be a solution.

Then $Hb^{t+1} = \alpha Hb^t \therefore b = \alpha, (b^t \neq 0, H \neq 0)$

So, $Y_t = Hb^t$ is our solution for the homogeneous part, i.e., $Y_c = H\alpha^t$.

Let us consider particular solution (Y_p).

Let $Y_t = K$ (a constant) be the particular solution.

As K is a constant, it holds for all t.

Now, $Y_{t+1} = \alpha Y_t - \beta$

$$\therefore K = \alpha K - \beta \quad \therefore K = -\frac{\beta}{1-\alpha}$$

So, general solution, $Y_t = Y_C + Y_p$ or, $Y_t = H\alpha^t - \frac{\beta}{1-\alpha}$

The value of H is to be determined from the initial condition, i.e., by putting t = 0.

$$\text{Then, } Y_0 = H - \frac{\beta}{1-\alpha}$$

$\therefore H = Y_0 + \frac{\beta}{1-\alpha}$. Putting this value of H, we get final solution,

$$Y_t = \left[Y_0 + \frac{\beta}{1-\alpha}\right]\alpha^t - \frac{\beta}{1-\alpha}$$

This is our time path of Y.

SOLUTION OF DIFFERENCE EQUATION BY ITERATIVE METHOD

A first order difference equation describes the pattern of change of a variable Y between two consecutive periods only. So, once such pattern is specified and once we are given the initial value Y_0, we can find Y_1 from the equation. Now, once Y_1 is known, we can determine Y_2 from the given pattern of equation just by putting the expression of Y_1 and so on. Thus, we can get the value of Y for any time period (t) just by repeated application (iteration) of the pattern of change specified in the difference equation. Hence the method is known as iterative method.

Example 6.4 : We take the example 6.1 considered in general method.

Solve $Y_t = 2Y_{t-1} + 3$ by iterative method.

Solution : We have, $Y_t = 2Y_{t-1} + 3$

Putting t = 1, $Y_1 = 2Y_0 + 3$

Now, putting t = 2, we get,

$Y_2 = 2Y_1 + 3 = 2(2Y_0 + 3) + 3 = 2^2Y_0 + 2^2.3 - 3 = 2^2(Y_0 + 3) - 3$

When t = 3, $Y_3 = 2Y_2 + 3 = 2[2^2(Y_0 + 3) - 3] + 3 = 2^3(Y_0 + 3) - 2 \times 3 + 3$

$= 2^3(Y_0 + 3) - 3$

Proceeding in this manner, we get, $Y_t = (Y_0 + 3)2^t - 3$.

This is our time path of Y. We got the same result in example 6.1 by following the general method.

Example 6.4. : Solve $Y_{t+1} = Y_t+1$ by iterative method when $Y_0 = 10$.

Solution : We have : $Y_{t+1} = Y_t+1$. From this, we can write, $Y_t = Y_{t-1} + 1$.

Now, putting t = 1, 2, 3, ..., we get,

If t = 1, $Y_1 = Y_0 + 1$

If t = 2, $Y_2 = Y_1 + 1 = (Y_0 + 1) + 1 = Y_0 + 2$

If t = 3, $Y_3 = Y_2 + 1 = (Y_0 + 2) + 1 = Y_0 + 3$

Thus, we get, $Y_t = Y_0 + t$. Given that $Y_0 = 10$. So, our time path is : $Y_t = 10 + t$, the same result obtained in example 6.2. through general method.

Example 6.5 : Solve the difference equation $Y_{t+1} = 0.5\ Y_t$ by iterative method.

Solution : We have, $Y_{t+1} = 0.5\ Y_t$. This is a first order homogeneous difference equation. Now putting t = 0, 1, 2, 3, ... etc. we get,

$Y_1 = 0.5\ Y_0$

$Y_2 = 0.5\ Y_1 = (0.5)^2.\ Y_0$

$Y_3 = 0.5\ Y_2 = 0.5(0.5)^2\ Y_0 = (0.5)^3.\ Y_0$

Thus, we get, $Y_t = Y_0(0.5)^t$. This is our desired solution.

Example 6.6 : Given $I_t = v(Y_t - Y_{t-1})$ and $S_t = sY_{t-1}$. Determine the equilibrium growth path of Y (income) where I = investment and S = saving.

Solution : In equilibrium, $I_t = S_t$ or, $v(Y_t - Y_{t-1}) = sY_{t-1}$

or, $\nu Y_t = \nu Y_{t-1} + sY_{t-1} = (\nu + s)Y_{t-1}$

$\therefore\ Y_t = \left(1+\frac{s}{\nu}\right)Y_{t-1}$. This is a first order homogeneous difference equation in Y. The solution of this equation will give us the time path of Y (income).

We have, $Y_t = (1 + s/\nu)Y_{t-1}$

If t = 1, we get, $Y_1 = (1 + s/\nu)Y_0$

If t = 2, we get, $Y_2 = (1 + s/\nu)Y_1 = (1 + s/\nu)(1 + s/\nu)Y_0 = (1 + s/\nu)^2 Y_0$

If t = 3, we get, $Y_3 = (1 + s/\nu)Y_2 = (1 + s/\nu)(1 + s/\nu)^2. Y_0 = (1 + s/\nu)^3 Y_0$

Proceeding in this manner, we finally get, $Y_t = Y_0(1 + s/\nu)^t$.

This is our time path of Y. We may do the same thing in an alternative manner.

When t = 1, $Y_1 = (1 + s/\nu)Y_0$

When t = 2, $Y_2 = (1 + s/\nu)Y_1$

When t = 3, $Y_3 = (1 + s/\nu)Y_2$

Putting t, $Y_t = (1 + s/\nu)\, Y_{t-1}$

Multiplying both sides, we get, $Y_1.Y_2.Y_3 ...Y_t = \left(1+\frac{s}{\nu}\right)^t . Y_0.Y_1.Y_2....Y_{t-1}$. Cancelling $Y_1, Y_2, ..., Y_{t-1}$ from both sides, we get, $Y_t = Y_0\left(1+\frac{s}{\nu}\right)^t$. This is our time path of Y.

$$\text{The rate of growth of Y} = \frac{Y_t - Y_{t-1}}{Y_{t-1}} = \frac{Y_0(1+s/\nu)^t - Y_0(1+s/\nu)^{t-1}}{Y_0(1+s/\nu)^{t-1}}$$

$$= \frac{Y_0(1+s/\nu)^{t-1}(1+s/\nu-1)}{Y_0(1+s/\nu)^{t-1}} = s/\nu$$

We may get the same result from our equilibrium condition, $I_t = S_t$

or, $\nu(Y_t - Y_{t-1}) = s.Y_{t-1}$

or, $\frac{(Y_t - Y_{t-1})}{Y_{t-1}} = s/\nu$ i.e., the rate of growth of Y = s/ν.

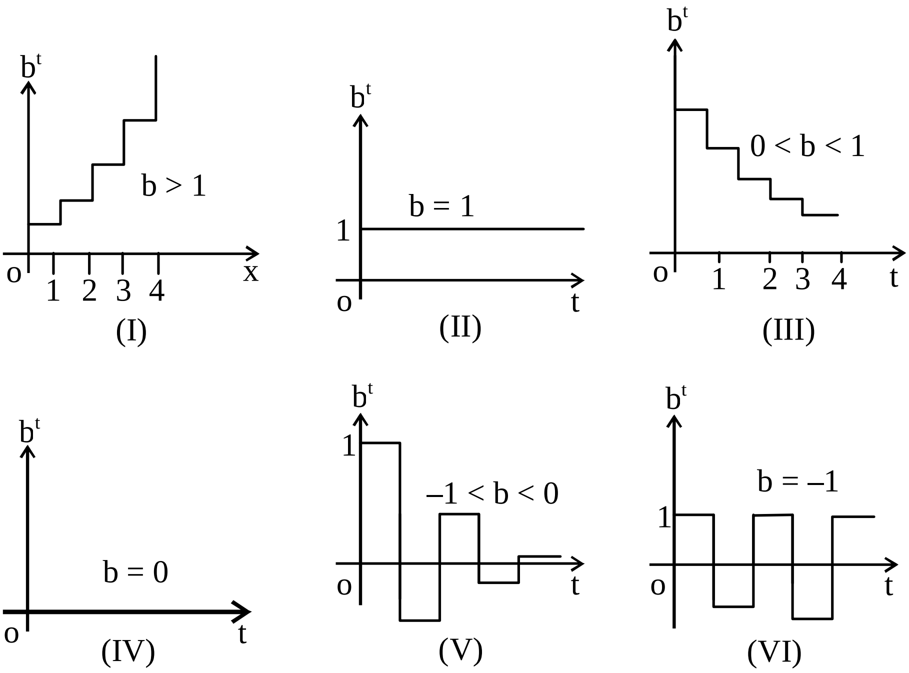
When a sum p grows by the rate r, the amount after t years is : $A = p(1 + r)^t$. Hence, when Y rises by the rate $\frac{s}{\nu}$, the value of Y after t years = $Y_t = Y_0\left(1+\frac{s}{\nu}\right)^t$ which is our time path of Y. [In our example, equations have been taken from Harrod's model of economic growth. There, $\frac{s}{\nu}$ is called the warranted rate of growth]

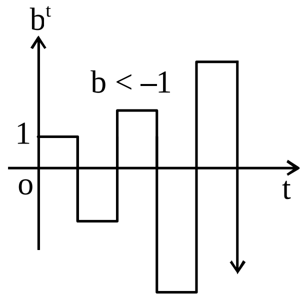
6.3.2 Dynamic Stability of Equilibrium

The equilibrium is dynamically stable if the complementary function tends to zero as $t \to \infty$. Now, in a first order difference equation, the complementary solution is : $Y_c = Hb^t$. We first consider the significance of b, ignoring the coefficient H(by assuming H = 1). For analytical purpose, we can divide the range of possible values of b into 7 regions.

Region	Value of b	Value of b^t	Value of b^t in different time periods				
			t = 0	t = 1	t = 2	t =3	t = 4
I	b > 1 (|b| > 1), e.g., 2^t		1	2	4	8	16
II	b = 1 (|b| = 1), i.e., 1^t		1	1	1	1	1
III	0 < b < 1 (|b| < 1), e.g., $\left(\frac{1}{2}\right)^t$		1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
IV	b = 0 (|b| = 0), e.g., $(0)^t$		0	0	0	0	0
V	–1 < b < 0 (|b| < 1), e.g., $\left(-\frac{1}{2}\right)^t$		1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$
VI	b = –1 (|b| = 1) i.e. $(–1)^t$		0	–1	1	–1	1
VII	b < –1 (|b| > 1) e.g., $(–2)^t$		1	–2	4	–8	16

The time path corresponding to different values of b are shown below :





(VII)

The essence of these figures can be summed up in the following statement :

The time path of b^t will be non-oscillatory if b > 0 and oscillatory if b < 0. On the other hand, it will be divergent if |b| > 1 and convergent if |b| < 1.

We give some examples.

Example 6.7 : What kind of time path is represented by $Y_t = 2\left(-\frac{4}{5}\right)^t + 9$?

Solution : If t = 0, Y_0 = 9. This is the equilibrium level of Y. Since $b = -\frac{4}{5} < 0$, the time path of Y is oscillatory. But since $|b| = \frac{4}{5} < 1$, the oscillation is damped, and the time path converges to the equilibrium level of 9.

Example 6.8 : Examine the nature of time path of $Y_t = 3(2)^t + 4$?

Solution : At t = 0, Y_0 = 4 = initial or equilibrium value of Y. Since b = 2 > 0, no oscillation will occur. But since |b| = 2 > 1, the time path will diverge from the equilibrium level of 4.

6.3.3 Solution of a Second Order Difference Equation

We shall consider the solution of a second order difference equation with constant term and constant coefficients. The second order non-homogeneous difference equation is : $aY_t + bY_{t-1} + cY_{t-2} + d = 0$

We first consider the solution of the homogeneous part, i.e., complementary solution (Y_c). Then our equation is :

$aY_t + bY_{t-1} + cY_{t-2} = 0$

Let $Y_t = x^t$ be a solution.

Then, $ax^t + bx^{t-1} + cx^{t-2} = 0$

$\therefore\ ax^2 + bx + c = 0$ (assuming $x^{t-2} \neq 0$)

Let us solve this equation. Multiplying both sides by 4a,

we get, $4a^2x^2 + 4abx + 4ac = 0$

or, $(2ax)^2 + 2.2ax.b + b^2 - b^2 + 4ac = 0$

or, $(2ax + b)^2 = b^2 - 4ac$

or, $2ax + b = \pm\sqrt{b^2 - 4ac}$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus the roots of our quadratic equation are : $(x_1, x_2) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So, the solutions are x_1^t and x_2^t. In this case, $Y_t = K_1x_1^t + K_2x_2^t$ is the solution. The values of K_1 and K_2 are to be determined from initial conditions. Two initial conditions are needed in the second-order case. Let, when t = 0, $Y = Y_0$ and when t = 1, $Y = Y_1$. Then, $K_1x_1^0 + K_2x_2^0 = Y_0$, i.e., $K_1 + K_2 = Y_0$ (given).

As x_1 and x_2 are already known, we can solve for K_1 and K_2. Thus, the complementary solution is : $Y_C = K_1x_1^t + K_2x_2^t$ where x_1 and x_2 are the two roots of the quadratic,

$ax^2 + bx + c = 0$ i.e., $(x_1, x_2) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac \geq 0$, or, $b^2 \geq 4ac$, roots will be real. If $b^2 < 4ac$, or, $b^2 - 4ac < 0$, roots will be complex. Then the solution involves the trigonometric functions sine and cosine. We here just state the solutions. We introduce the following notations :

$$\nu_1 = -\frac{b}{2a} \text{ and } \nu_2 = -\frac{b^2 - 4ac}{2a},\ R = \sqrt{\nu_1^2 + \nu_2^2}$$

Then we have to find the angle z, the sine of which is $\dfrac{\nu_2}{\sqrt{\nu_1^2 + \nu_2^2}}$ and the cosine of which is $\dfrac{\nu_1}{\sqrt{\nu_1^2 + \nu_2^2}}$.

Then the solution is :

$Y_t = R^t [w_1 \sin(tz) + w_2 \cos(tz)]$

where w_1 and w_2 are constants to be determined from the two initial conditions (i.e., for t = 0 and t = 1).

Let us consider the particular solution (Y_p).

Let Y_t = K (constant) be a solution. So, aK + bK + cK + d = 0

$$\therefore K = -\frac{d}{a+b+c} \text{ provided } (a + b + c) \neq 0.$$

Then the general solution or the complete solution is,

$$Y_t = K_1 x_1^t + K_2 x_2^t + \frac{-d}{a+b+c} \text{ provided } (a + b + c) \neq 0.$$

If (a + b + c) = 0, we assume Y_t = Kt as the solution.

Then aKt + bK(t – 1) + cK(t – 2) + d = 0

or, aKt + bKt + cKt – bK – 2cK + d = 0

or, Kt(a + b + c) – bK – 2cK + d = 0

or, K(–b – 2c) = – d (as a + b + c = 0)

$$\therefore K = \frac{-d}{-b-2c}, \text{ provided } (-b - 2c) \neq 0.$$

Then the solution is, $Y_t = K_1 x_1^t + K_2 x_2^t + Kt$, provided (–b – 2c) ≠ 0.

If (–b – 2c) = 0, we should take $Y_t = Kt^2$ as the particular solution and proceed in the same manner. In the first order case, we see that either Y_t = K or, Y_t = Kt leads to correct particular solution. In the second order case, either Y_t = K, or Y_t = Kt, or $Y_t = Kt^2$ leads to the correct particular solution.

6.4 Differential Equation

We know that in static analysis, time is not considered as a separate variable. But in dynamic analysis, time is considered as a separate variable and the change in various variables is considered over time. Now, time may be treated as a discrete variable or as a continuous variable. If time is taken as a discrete variable, we use difference equation to deduce the time path of any variable. On the other hand, if time is considered as continuous, then to deduce the time path of any variable, we use differential equation. In a differential equation, a variable is taken as a continuously differentiable function of time.

6.4.1 Solution of a First Order Differential Equation

We shall consider the first order linear differential equation with constant coefficient and constant term. The general form of a first order differential equation is :

$\frac{dy}{dt}+u(t)y=w(t)$ where u and w are two functions of t(time), as is y. If u is a constant and w is a constant additive term, we get a first order linear differential equation with constant coefficient and constant term. Let u = a and w = b. Then we have, $\frac{dy}{dt}+ay=b$.

It is a first order non-homogeneous differential equation. Again, if b = 0, we have, $\frac{dy}{dt}+ay=0$. Then the function is homogeneous and if $b \neq 0$, the function is said to be non-homogeneous.

Solution in homogeneous case

The equation of a linear homogeneous differential equation is : $\frac{dy}{dt}+ay=0$.

We shall consider the solution of this equation, or, more specifically, we shall try to derive the time path of y.

From the above equation, we can write, $\frac{dy}{dt}=-ay$, or $\frac{1}{y}\frac{dy}{dt}=-a$

or, $\frac{dy}{y}=-a\ dt$.

Integrating we get, $\int\frac{dy}{y}=-a\int dt$

or, $\log_e y = -at + c$ where c = constant of integration.

$\therefore\ y = e^{-at+c} = e^c.e^{-at}$

or, $y = H\ e^{-at}$ where $H = e^c$.

Thus, $y(t) = H\ e^{-at}$ is the general solution of the given differential equation. The value of H is to be determined from initial condition i.e., by putting t = 0. Then y(0) = H. Thus, the definite solution is : $y(t) = y(0)e^{-at}$.

Two things should be noted about the solution of a differential equation :

(i) The solution is not a numerical value here, but a function of t. If t is time, we get a time path.

(ii) The solution y(t) is free of any derivative or differential expression. Hence, as soon as a specific value of t is substituted into it, a corresponding value of y can be calculated directly.

Solution in non-homogeneous case

The general form of a non-homogeneous linear differential equation is : $\frac{dy}{dt} + ay = b$.

Here the general solution will have two parts :

(i) Homogeneous solution or complementary solution (y_c)

(ii) Particular solution (y_p)

i.e., $y(t) = y_c + y_p$.

For the homogeneous solution, we take the equation in homogeneous form : $\frac{dy}{dt} + ay = 0$

Then we have seen that the homogeneous solution is : $y_c = H\,e^{-at}$

Let us consider particular solution. For particular solution, we assume, y = k, (k = constant). Then $\frac{dy}{dt} = 0$

So, 0 + ay = b, or, ak = b $\quad \therefore k = \frac{b}{a}$. Thus, $y_p = \frac{b}{a}$, $(a \neq 0)$.

Thus, the general solution or complete solution is :

$y(t) = y_c + y_p = H\,e^{-at} + \frac{b}{a}$, $(a \neq 0)$.

The value of H is to be determined from the initial condition, i.e., by putting t = 0.

Then, $y(0) = H + \frac{b}{a}$, or, $H = y(0) - \frac{b}{a}$

Hence the definite or final solution is :

$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$, $(a \neq 0)$.

Example 6.9 : Given $\frac{dy}{dt} + 2y = 6$ with the initial condition y(0) = 10. Solve the equation or deduce the time path of y.

Solution : Here a = 2, b = 6 in the equation, $\frac{dy}{dt} + ay = b$.

In this case, the solution is, $y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$.

Putting the value in this formula, we get, $y(t) = \left[10 - \frac{6}{2}\right]e^{-2t} + \frac{6}{2}$

or, $y(t) = 7e^{-2t} + 3$ **(Ans.)**

Solution without formula

Our given differential equation is : $\frac{dy}{dt} + 2y = 6$

It is a first order non-homogeneous difference equation. First we consider the solution of the homogeneous part.

Then, $\frac{dy}{dt} + 2y = 0$.

or, $\frac{dy}{dt} = -2y$, or, $\frac{dy}{y} = -2dt$

Integrating, we get, $\log_e y = -2t + c$ where c = constant

or, $y = e^{-2t+c} = e^c.e^{-2t}$

$\therefore$ $y = He^{-2t}$ where $H = e^c$. This is the solution for the homogenous part (y_c).

We now consider the particular solution, y_p.

Let y = k, a constant, be the particular solution. Then $\frac{dy}{dt} = 0$.

So, from the given equation, $\frac{dy}{dt} + 2y = 6$, we get, $0 + 2y = 6$ $\therefore$ $y = 3$, or, $k = 3$.

This is our particular solution (y_p) i.e., $y_p = 3$.

So, general solution, $y(t) = y_c + y_p = He^{-2t} + 3$ where the value of H is to be determined from the intial condition i.e., by putting t = 0.

$\therefore$ $y(0) = He^0 + 3 = H + 3$ $\therefore$ $H = y(0) - 3$

We are given that y(0) = 10. $\therefore$ $H = y(0) - 3 = 10 - 3 = 7$

Hence the final solution is : $y(t) = 7e^{-2t} + 3$ **(Ans.)**

Example 6.10 : Solve the equation $\frac{dy}{dt} + 4y = 0$ with the initial condition y(0) = 1.

Solution : For a linear homogeneous differential equation $\frac{dy}{dt} + ay = 0$, the solution is : $y(t) = y(0)e^{-at}$

Here, in our given problem, a = 4 and y(0) = 1.

Putting these values in our formula, we get, $y(t) = 1.e^{-4t}$ or, $y(t) = e^{-4t}$ **(Ans.)**

Solution without applying formula

We have, $\frac{dy}{dt} = -4y \quad \therefore \quad \frac{dy}{y} = -4dt$

Integrating, $\log_e y = -4t + c$ where c = constant of integration

$\therefore \; y(t) = e^{-4t+c}$

or, $y(t) = e^c.e^{-4t} = He^{-4t}$ where $H = e^c$.

The value of H is to be determined from the initial condition, i.e., by putting t = 0.

$\therefore \; y(0) = He^0 = H$.

So, H = y(0) = 1 (given in the problem).

So, $y(t) = He^{-4t} = e^{-4t}$ is the required solution.

Example 6.11 : Solve the equation $\frac{dy}{dt} = b$

Solution : we have, $\frac{dy}{dt} = b \quad \therefore \; dy = b.dt$

Integrating, y(t) = bt + c where c is a constant. Its value will be known from the initial condition, i.e., t = 0.

Then $y(0) = b \times 0 + c \quad \therefore \; c = y(0)$.

Therefore, the solution is : y(t) = y(0) + bt **(Ans.)**

Alternative method : The general form of a first order differential equation is : $\frac{dy}{dt} + ay = b$.

In our given problem, $\frac{dy}{dt} = b \quad \therefore$ a = 0. In this case, the complementary solution (y_c) is : $y(t) = He^{-at}$.

It t = 0, $y(0) = He^o = H$ where H is an arbitrary constant.

Let us consider the particular solution, y_p.

Let y = K, a constant, be a solution. Then, $\frac{dy}{dt} = 0$.

But we are given that $\frac{dy}{dt} = b$. So, we should try another particular solution.

Let y = Kt be a solution.

Then $\frac{dy}{dt} = K$. But we are given that $\frac{dy}{dt} = b$

$\therefore$ K = b. So, our particular solution is : $y_p = Kt = bt$.

Hence, the general solution is : $y(t) = y_c + y_p = H + bt$

where the value of H is to be determined from the initial condition, i.e., by putting t = 0. So, y(0) = H.

Hence the definite or total solution is : y(t) = y(0) + bt

Example 6.12 : Solve the equation $\frac{dy}{dt} = 2$ with the initial condition y(0) = 5.

Solution by formula : We know that for differential equation $\frac{dy}{dt} = b$, the solution is : y(t) = y(0) + bt (we have seen it in our previous example).

In our given problem, b = 2 and y(0) = 5.

So, the required solution is : y(t) = 5 + 2t **(Ans.)**

[or we may follow the alternative method as shown in the preveious example].

6.5. Application of Difference Equation in Economics

Difference equation has many applications in economics. It is used to determine the time path of an economic variable, to examine the stability of an equilibrium over time to determine the value of a variable after some given periods, etc. In the present section, we shall consider the application of difference equation in three particular cases : (i) in the context of Keynesian dynamic multiplier, (ii) price stability in a cobweb model and (iii) interaction between multiplier and accelerator as a possible explanation of the emergence of trade cycles.

Let us consider them one by one.

6.5.1 Keynesian Dynamic Multiplier

We introduce dynamic element into the simple Keynesian model. We assume that consumption expenditure in period t is a linear function of income of the previous period i.e., $C_t = f(Y_{t-1})$. In particular, we assume that $C_t = bY_{t-1} + a$, $0 < b < 1$, $a > 0$. That is, there is one period lag in consumption function.

Investment is assumed to be autonomously given, i.e., $I_t = A$ where $A > 0$. The condition of equilibrium in the commodity market is : Aggregate supply in period t = aggregate demand in period t. We assume a closed economy with no economic activities on the part of the government. So, aggregate demand will have two components : C_t and I_t. Thus, our equilibrium condition is : $Y_t = C_t + I_t$.

Hence, in our simple Keynesian model, we have the following equations :

(1) $C_t = a + b\,Y_{t-1}$, $0 < b < 1$, $a > 0$ (autonomous consumption)

(2) $I_t = A$ (autonomous investment)

(3) $Y_t = C_t + I_t$ (Equilibrium condition)

Putting the values of C_t and I_t into the equilibrium condition, we get,

$Y_t = bY_{t-1} + (a + A)$.

or, (4)$Y_t = bY_{t-1} + z$ where $z = a + A$ = aggregate autonomous expenditure.

Equation (4) is a first order non-homogeneous difference equation. In order to solve this equation, we first consider the complementary solution (Y_c). To do this we take the homogeneous part : $Y_t = bY_{t-1}$. Let $Y_t = Hx^t$ be a solution for this equation. Hence, from the homogeneous part $Y_t = b\,Y_{t-1}$, we get, $Hx^t = bHx^{t-1}$

$\therefore\ x = b$, (assuming $H \neq 0$ and $x^{t-1} \neq 0$).

So, $Y_c = Hb^t$ is our complementary solution.

Let $Y_t = K$, a constant, be the particular solution.

Then this value of Y will hold for all t. So, we get, putting $Y_t = Y_{t-1} = K$,

$Y_t = bY_{t-1} + z$

or, $K = bK + z \ \therefore\ K = \frac{z}{1-b}$, $(b \neq 1)$

So, $Y_p = \frac{z}{1-b}$ is the particular solution. This is actually equilibrium value of Y.

Now, the general solution is : $Y_t = Y_c + Y_p$

or, $Y_t = Hb^t + \frac{z}{1-b}$, $(b \neq 1)$

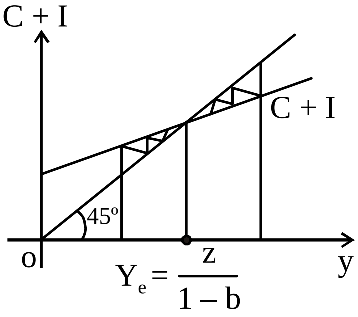
The value of H is to be determined from the initial condition, i.e., by putting $t = 0$. Then we get, $Y_0 = H + \frac{z}{1-b}\ \therefore\ H = Y_0 - \frac{z}{1-b}$.

Hence, our definite solution is : $Y_t = Hb^t + \frac{z}{1-b}$

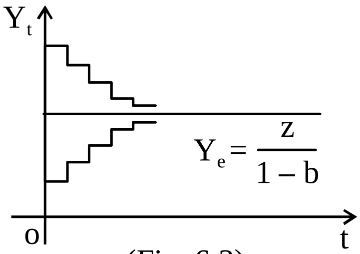
or, $$Y_t = \left[Y_0 - \frac{z}{1-b}\right]b^t + \frac{z}{1-b}$$

This is our time path of Y(income). Given $0 < b < 1$, as $t \to \propto$, $b^t \to 0$.

So, $Y_t \to \frac{z}{1-b}$. That is, Y_t tends to the equilibrium value. Thus our equilibrium is stable if $0 < b < 1$. We have shown it in our figures 6.2 and 6.3 below. If the value of Y is less than the equilibrium value, then aggregate demand $(C_t + I_t)$ > aggregate supply i.e., there is excess demand. So, the levle of output (Y_t) will rise.



(Fig. 6.2)



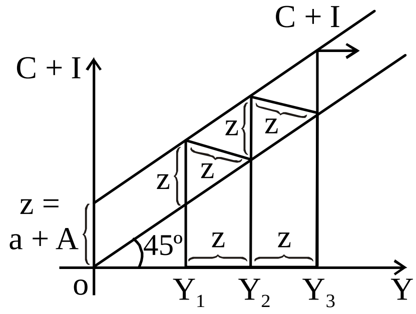
(Fig. 6.3)

If the value of Y is greater than the equilibrium value then aggregate demand $(C_t + I_t)$ < aggregate supply i.e., there will be excess supply. So, the level of output (Y_t) will fall. Thus, our equilibrium is stable provided $0 < b < 1$.

This is shown in our two figures.

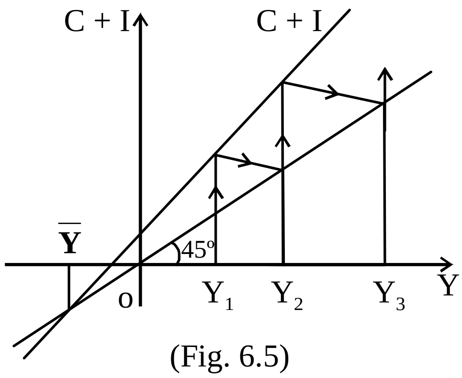
If b = 1, $Y_t = Y_{t-1} + z$. Here, as before, $Y_c = Hb^t = H$ (as b = 1)

Let us consider particular solution. Let $Y_t = K$ be the particular solution. So, this value will hold for all t. Hence we get, K = K + z, or, z = 0. But, we know that $z \neq 0$. Hence, $Y_t = K$ will not be the particular solution. Hence we try another. Let $Y_t = Kt$ be the particular solution. Then we have, Kt = K(t – 1) + z. or, K = z. So, the particular solution, $Y_p = Kt = zt$. Thus, total solution, $Y_t = Y_c + Y_p = H + zt$. The value of H will be determined from the initial condition (i.e., by putting t = 0). If t = 0, $H = Y_0$. So, we have, $Y_t = Y_0 + zt = Y_0 + (a + A)$. This means that the level of income (Y) will go on increasing by the amount of aggregate autonomous expenditure (z = a + A) every time if b = 1. In static multiplier, $\frac{1}{1-b} = \propto$ if b = 1. Then, as autonomous investment rises, Y immediately jumps to infinity. This is absurd. This absurdity can be easily explained by dynamic analysis. Here we say that when b = 1, Y does not jump to inifinity immediately. Rather, here Y rises everytime by z (= a + A). So, when $t \rightarrow \mu$, Y tends to infinity. This is shown in the figure 6.4.



(Fig. 6.4)

Again, if b > 1, the static multiplier is negative. $\left(\frac{1}{1-b} < 0\right)$. This means that as



(Fig. 6.5)

autonomous expenditure rises, equilibrium level of income falls. This is again absurd. This absurdity can be removed by dynamic analysis. If b > 1, the C + I curve is steeper than the 45° line and the equilibrium level of income is negative. Thus, here the equilibrium does not exist. For any positive output, aggregate demand ($C_t + I_t$) > aggregate supply (Y_t). So, the level of income (Y) goes on increasing indefinitely. This is shown in the figure 6.5 where we get an explosive situation.

6.5.2 Cobweb Model

Let $D_t = \alpha - \beta p_t$, $S = -\gamma + \delta p_{t-1}$, $(\alpha, \beta, \gamma, \delta > 0)$.

In equilibrium, demand = supply i.e., $D_t = S_t$.

$\therefore\ \alpha - \beta p_t = -\gamma + \delta p_{t-1}$

or, $\beta p_t + \delta p_{t-1} = \alpha + \gamma$

or, $p_t + \frac{\delta}{\beta} \cdot p_{t-1} = \frac{\delta + \beta}{\beta}$ or, $p_{t+1} + \frac{\delta}{\beta} p_t = \frac{\delta+\beta}{\beta}$

This is a first order non-homogeneous difference equation of the form : $Y_{t+1} + aY_t = c$.

In this case, the time path is : $Y_t \left[Y_0 - \frac{c}{1+a}\right](-a)^t + \frac{c}{1+a}$.

Here Y = p, $a = \frac{\delta}{\beta}$ and $c = \frac{\alpha+\gamma}{\beta}$

$$\therefore\ \frac{c}{1+a} = \frac{\frac{\alpha+\beta}{\gamma}}{1+\frac{\delta}{\beta}} = \frac{\alpha+\gamma}{\beta} \times \frac{\beta}{\beta+\delta} = \frac{\alpha+\gamma}{\beta+\delta}$$

So, $p_t = \left(p_0 - \frac{\alpha+\gamma}{\beta+\delta}\right)\left(-\frac{\delta}{\beta}\right)^t + \frac{\alpha+\gamma}{\beta+\delta}$ where p_0 represents the initial price. Further, the particular solution is obtained by putting $p_{t+1} = p_t = \overline{p}$ (say).

Then, $\overline{p} + \frac{\delta}{\beta}.\overline{p} = \frac{\alpha+\gamma}{\beta}$ or, $\overline{p}\left(1 + \frac{\delta}{\beta}\right) = \frac{\alpha+\gamma}{\beta}$ or, $\overline{p}\left(\frac{\beta+\delta}{\beta}\right) = \frac{\alpha+\gamma}{\beta}$.

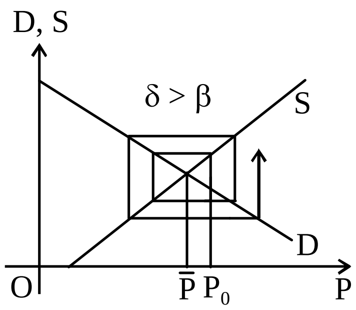
So, $\bar{p} = \frac{\alpha+\gamma}{\beta} \times \frac{\beta}{\beta+\delta} = \frac{\alpha+\gamma}{\beta+\delta}$.

This particular solution $\frac{\alpha+\gamma}{\beta+\delta}$ is the equilibrium price. We denote it by $\bar{p}$. So the time path of p is : $p_t = (p_0 - \bar{p})\ \ -^{t}\ \ \bar{p}$

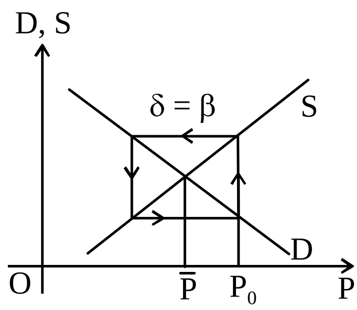
The nature of time path of price will depend on the sign and value of $\left(-\frac{\delta}{\beta}\right)$. As δ and β are positive, $\left(-\frac{\delta}{\beta}\right) < 0$. Hence, $-^{t}$ will be negative when t is odd and it is positive when t is even. So, our time path will be oscillatory. The oscillation will be explosive, uniform or damped according as $\delta \gtreqless \beta$. These cases are shown in the figures. In the figure 6.6, we have shown the oscillation when $\delta > \beta$. In this case, there is explosive oscillation in price.

In the figure 6.7, we have shown the case where $\delta = \beta$. In this case, the price will have constant oscillation.

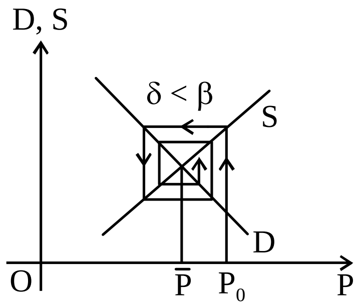
In the figure 6.8, we have shown the case when $\delta < \beta$. In this case, p will have a damped oscillation.



(Fig. 6.6)



(Fig. 6.7)



(Fig. 6.8)

In this case, p will ultimately converge to the equilibrium value (however, technically that value will be reached only after infinity period). We shall say that the equilibrium is stable in the sense that the actual price will gradually move towards the equilibrium value.

6.5.3 Multiplier-Accelerator Model of Trade Cycle (Samuelson's Model of Business Cycle)

There are so many theories to explain business cycles. They may be divided into two groups–non monetary theories and monetary theories. Schumpeter has given a non-monetary theory of trade cycle in terms of innovations while Pigou has given a psychological theory. Another non-monetary theory is the climatic theory of Jevons. Among the monetary theories, the most important is the Hawtrey theory which seeks to explain business cycle in terms of expansion or contraction in bank credit.

Samuelson has given a non-monetary theory of trade cycle. He argues that trade cycles are created due to the interaction between the multiplier and the accelerator. Later, Hicks further developed this theory. We shall briefly discuss Samuelson's Multiplier – Accelerator theory.

From the multiplier theory, we know that when there is an increase in autonomous expenditure, the equilibrium level of income rises by a multiplier effect. Again, from the Acceleration Principle, we know that a change in the level of income will bring a change induced investment. This will again lead to a change in the level of income through multiplier process. Thus, there is an interaction between the multiplier and the accelerator.

According to Samuelson, the interaction between multiplier and accelerator creates cyclical fluctuations in income. To show it, we consider a model which is based on the following assumptions :

(1) There are two groups in the economy – households and firms. So, aggregate demand = $C_t + I_t$.

(2) The consumption function is assumed to be linear. There is one period lag in the consumption function. $C_t = bY_{t-1}$ $(0 < b < 1)$

(3) The investment function is given by the acceleration principle. There is one period lag in the investment function. So, $I_t = v(Y_{t-1} - Y_{t-2})$ where v is the accelerator.

Now, equilibrium requires, aggregate supply of goods and services = aggregate demand for goods and services, i.e., $Y_t = C_t + I_t$

or, $Y_t = bY_{t-1} + v(Y_{t-1} - Y_{t-2})$

or, $Y_t - (b + v)Y_{t-1} + vY_{t-2} = 0$

Putting b (= MPC) = 1 – s i.e., 1 – MPS, we get, $Y_t - (1 - s + v)Y_{t-1} + vY_{t-2} = 0$

This is a second order homogeneous difference equation. The solution of this equation gives us the time path of income.

Let $Y_t = x^t$ be a solution. Then, we may write, $x^t - (1 - s + v)x^{t-1} + vx^{t-2} = 0$

or, $x^2 - (1 - s + v)x + v = 0$, (assuming $x^{t-2} \neq 0$).

This is a quadratic equation in x. Let x_1 and x_2 be the roots of this equation. Then,

$$(x_1, x_2) = \frac{(1-s+v) \pm \sqrt{(1-s+v)^2 - 4v}}{2}.$$

The solution of the difference equation can be written as, $Y_t = \beta_1 x_1^t + \beta_2 x_2^t$

where β_1 and β_2 are two arbitrary constants to be determined from the initial condition. The nature of the time path of income depends on the nature of the roots x_1 and x_2.

The sum of the roots, $x_1 + x_2 = 1 - s + v > 0$.

The product of the roots, $x_1 x_2 = v > 0$

So, both the roots are positive.

Now, roots will be real if $(1 - s + v) \geq 4v$

or, $(1 - s + v) \geq 2\sqrt{v}$, or, $(1 - 2\sqrt{v} + v) \geq s$

So, (i) $(1-\sqrt{v})^2 \geq (\sqrt{s})^2$ and (ii) $(\sqrt{v}-1)^2 \geq (\sqrt{s})^2$

From (i), we get, $(1-\sqrt{v}) > \sqrt{s}$ or, $\sqrt{v} \leq 1-\sqrt{s}$ or, $v \leq (1-\sqrt{s})^2$

From (ii), $(\sqrt{v}-1) \geq \sqrt{s}$ or, $\sqrt{v} \geq 1+\sqrt{s}$ or $v \geq (1+\sqrt{s})^2$

Case (i) : When $v \leq (1-\sqrt{s})^2$, i.e., v is less than unity, both x_1 and x_2 are less than unity. Then $x_1^t \to 0$ and $x_2^t \to 0$ as $t \to \infty$. Thus, there will be steady decline in the level of income.

Case (ii) : When $v \geq (1+\sqrt{s})^2$, i.e., $v > 1$, then at least one of the roots is greater than unity. So, either x_1^t or x_2^t or both will tend to ∞ as $t \to \infty$. In this case, as time passes on, there will be steady growth in income.

Roots will be complex if $(1 - s + v)^2 < 4v$. Proceeding as before, we get the following condition : $(1-\sqrt{s})^2 < v < (1+\sqrt{s})^2$.

Here we get 3 cases.

Case (iii) : $(1-\sqrt{s})^2 < v < 1$. Here v is less than unity. In this case, there will be fluctuations in the level of income and the fluctuations will be damped.

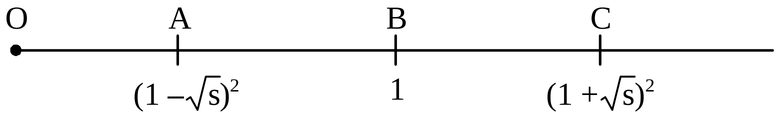
Case (iv) : . Here v is greater than unity. There will be fluctuations in the level of income and fluctuations will be explosive.

Case (v) : v = 1. In this case, there will be regular fluctuations in income.

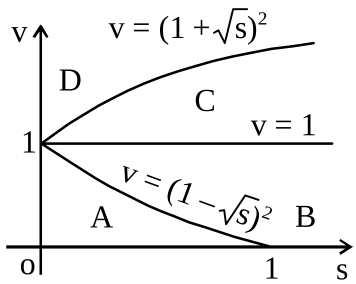
Thus we get different types of time path of income depending on the values of the parameters – s and v. In short, we can put our main results in the following table :

Range of v	Path of output or income
$v \leq (1-\sqrt{s})^2$	steady decline
$(1-\sqrt{s})^2 < v < 1$	damped oscillations
v = 1	regular oscillations
$1 < v < (1+\sqrt{s})^2$	explosive oscillations
$v \geq (1+\sqrt{s})^2$	steady growth

The different cases can also be represented in the following way.



Suppose the length OB = 1 unit. OA represents $(1-\sqrt{s})^2$ and OC represents $(1+\sqrt{s})^2$. Thus, if the value of v lies in the range OA, there will be steady decline and if v lies to the right of C, there will be steady growth. There will be oscillations if v lies in the range AC. The oscillations will be damped in the range AB and explosive in the range BC and regular at B.



(Fig. 6.9)

Thus, we see that time path of income depends on the values of s and v. For some combinations of values, there will be steady growth; for some combinations there will be steady decline; for some combinations there will be oscillations, etc. The various regions are shown in the figure 6.9 where we plot v on the vertical axis and s horizontal axis.

We have plotted three functions : $v = (1-\sqrt{s})^2$, v = 1 and $v < (1-\sqrt{s})^2$. In region A, we have $v < (1-\sqrt{s})^2$. In this region, there will be steady decline. In region D, we have $v < (1+\sqrt{s})^2$. In this region, there will be steady growth. In reagion B, we have $1 > v > (1-\sqrt{s})^2$ and there will be damped oscillations. In region C, we have

$1 < v < (1+\sqrt{s})^2$ and there will be explosive oscillations. On the line v = 1, there will be oscillations with constant amplitude. The oscillations take place in regions B and C where v lies between $(1-\sqrt{s})^2$ and $(1+\sqrt{s})^2$, that is, in cases (iii) and (iv) where the roots x_1 and x_2 are imaginary.

The above model cannot be, however, used as a satisfactory model of trade cycle. In the real world, trade cycles are more or less regular. But in Samuelson's model, cycles are regular if v = 1 which is a special case.

Secondly, in Samuelson's models, trade cycles will be symmetric. But such a symmetry is absent in the real world where depressions are generally shorter than booms.

However, this theory can be used as a useful ingredient of trade cycle theory. Several models have been built up on the basis of this model. Hence it occupies an important place in the theory of trade cycle. For example, Hicks has developed a more satisfactory theory of trade cycle on the basis of this multiplier-accelerator model.

6.6 Application of Differential Equation in Economics

We know that differential equations are used in any economic model using time as a continuous variable. In economics, we are interested to know the rate of change of various economic variables over time like price, output, investment, etc. Such rates of change can be known by applying differential equation. Hence there are so many applications of differential equations in Economics. In this section we shall consider two important applications of differential equation in the context of (i) Domar model of economic growth and (ii) Price dynamics in a competitive model. Let us discuss them one by one.

6.6.1 Domar Model of Economic Growth

Domar has tried to find out the condition for steady state growth in a free capitalist economy. He argues that investment has two roles. On the one hand, it raises aggregate demand. On the other hand, it raises productive capacity i.e., potential output. To have balance between demand and supply, investment should grow at a particular rate. Then there will be steady state growth in the economy. Otherwise, the economy will deviate from that steady state or equilibrium growth path. Let us consider the Domar model of economic growth.

Domar mentions that investment has two effects. **First**, investment raises aggregate demand through multiplier effect i.e., $\frac{dY}{dt} = \frac{1}{s}.\frac{dI}{dt}$. We get this in the following way. Equilibrium requires equality between planned saving and planned investment, i.e., S = I. It is assumed that S = s.Y, (0 < s < 1). So, we get, sY = I or, $Y = \frac{1}{s}.I$.

It is assumed that S, I and Y all are functions of time(t). So, differentiating both sides of our equation, we get, $\frac{dY}{dt} = \frac{1}{s}.\frac{dI}{dt}$. This is our multiplier effect. **Secondly**, investment raises or adds to productive capacity of an economy. Let P be the potential output and β be the capacity capital-output ratio, i.e., $\frac{P}{K} = \beta$ or, P = βK. Assuming P and K as functions of time (t), we can get $\frac{dP}{dt} = \beta.\frac{dK}{dt} = \beta.I$. So, I adds to productive capacity.

Now, in equilibrium, productive capacity of the economy is to be fully utilised i.e., Y = P. We assume that initially there is equilibrium in the economy. So, Y = P. Now, this equilibrium will be maintained in the next periods if $\frac{dY}{dt} = \frac{dP}{dt}$

or, $\frac{1}{s}.\frac{dI}{dt} = \beta I$. or, $\frac{1}{I}.\frac{dI}{dt} = s\beta$, i.e., if investment grows at the rate, s□β.

We can find out the time path of investment. Here, $\frac{dI}{dt} = s\beta I$, or, $\frac{dI}{dt} - s\beta I = 0$.

This is a first order linear homogeneous differential equation. Its solution is :
$I(t) = I(0)\, e^{-(-s\beta)t} = I(0)e^{s\beta t}$, where I(0) is the initial investment.
Clearly, the rate of growth of investment required for equilibrium is sβ.

This is actually warranted rate of growth (s/v) of Harrod : $s\beta = s.\frac{Y}{K} = \frac{s}{K/Y} = \frac{s}{v}$.

Now, what happens if the actual rate of growth(r) is different from sβ? If the actual rate of growth is r, then $I(t) = I(0)e^{rt}$

Then, $\frac{dI(t)}{dt} = r.I(0)e^{rt}$

So, $\frac{dY}{dt} = \frac{1}{s}.\frac{dI}{dt} = \frac{r}{s}.I(0)e^{rt}$. This is the rate of change of demand for output.

Again, from capacity or supply side.

$$\frac{dP}{dt} = \beta I(t) = \beta I(0)e^{rt} \quad \therefore \quad \frac{dY/dt}{dP/dt} = \frac{\frac{r}{s}.I(0)e^{rt}}{\beta.I(0)e^{rt}} = \frac{r}{s\beta}$$

If r > sβ i.e., $\frac{r}{s\beta} > 1$, then $\frac{dY}{dt} > \frac{dP}{dt}$.

That is, demand rises at a greater rate than the productive capacity. So, there will be excess demand and producers will further raise the actual rate of investment (r). Then r will further diverge from sβ.

Similarly, if r < sβ, $\frac{r}{s\beta} < 1$ and $\frac{dY}{dt} < \frac{dP}{dt}$ i.e., there will be excess capacity. So the producers will reduce the actual rate of investment(r). Then, again, r will diverge from sβ. The producers are making the wrong kind of adjustment. This is, in Harrodian jargon, known as knife edge instability. To maintain equilibrium, investment should grow only at the rate r = sβ. Any deviation from such a razor's edge will lead to either excess capacity or excess demand. Then the economy will deviate farther and farther from the equilibrium growth path. Hence the problem is popularly called the knife-edge problem.

6.6.2 Price Dynamics in a Competitive Model

In a competitive market, price is determined by the interplay of demand and supply. Let quantity demanded,

$q_d = \alpha - \beta p \ (\alpha, \beta > 0)$ ...(1)

and quantity supplied be

$q_s = -\gamma + \delta p \ (\gamma, \delta > 0)$ ...(2)

Further, $\frac{dp}{dt} = \theta(q_d - q_s), \ (\theta > 0)$ ...(3)

Equation (3) implies that the rate of change of price over time is directly proportional to the excess demand. We have to find out the equilibrium price and time path of price(p).

The equilibrium price $(\bar{p})$ can be obtained from demand - supply equality, i.e. putting $q_s = q_d$

or, $-\gamma + \delta\bar{p} = \alpha - \beta\bar{p}$. or, $(\beta + \delta)\bar{p} = \alpha + \gamma$

$\therefore \ \bar{p} = \frac{\alpha+\gamma}{\beta+\delta}$ ...(4)

This is our equilibrium price.

Let us consider the time path of price. Substituting (1) and (2) in (3), we get,

$$\frac{dp}{dt} = \theta[(\alpha - \gamma p) - (-\gamma + \delta p)]$$

$$\therefore \frac{dp}{dt} = \theta[(\alpha + \gamma) - (\beta + \delta)p]$$

Now, from equilibrium price given by equation (4),

$$\alpha + \gamma = (\beta + \delta)\bar{p} \quad \therefore \frac{dp}{dt} = \theta[(\beta + \delta)\bar{p} - (\beta + \delta)p]$$

$$\text{or, } \frac{dp}{dt} = \theta(\beta + \delta)(\bar{p} - p)$$

$$\text{or, } \frac{dp}{dt} + \theta(\beta + \delta)p = \theta(\beta + \delta)\bar{p}.$$

Putting $\theta(\beta + \delta) = k$, some constant, we get,

$$\frac{dp}{dt} + kp = k\bar{p}.$$

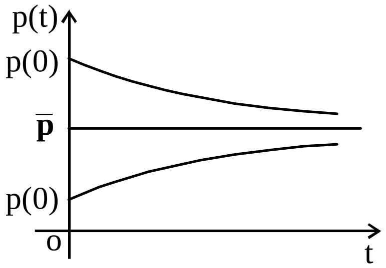
This is of the standard differential equation of the form : $\frac{dy}{dt} + ay = b$

Then the time path of y is : $y_t = y_c + y_p$

$$\text{or, } y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$$

So, in our context, $p(t) = \left[p(0) - \frac{k\bar{p}}{k}\right]e^{-kt} + \frac{k\bar{p}}{k}$ as $a = k$ and $b = k\bar{p}$.

Thus, we get, $p(t) = [p(0) - \bar{p}]e^{-kt} + \bar{p}$ or, $p(t) = [p(0) - \bar{p}]e^{-(\beta + \delta)t} + \bar{p}$.



(Fig. 6.10)

This is our desired time path of price. The first term on the R.H.S. is the complementary solution and the second term is the particular solution. As $t \to \infty$, $e^{-(\beta + \delta)t} \to 0$. Hence $p(t) \to \bar{p}$. This ($\bar{p}$) is the long run equilibrium price. If $p(0) > \bar{p}$, the time path of p will approach $\bar{p}$ from above (fig. 6.10). If $p(0) < \bar{p}$, the time path of p will approach $\bar{p}$ from below. This is shown in the figure. Here our equilibrium is stable, as p approaches long run equilibrium price.

We should note that here particular solution gives the equilibrium price and the complementary solution gives the deviations from equilibrium price.

We give below an example on price dynamics in a competitive model.

Example 6.13 : Demand and supply functions are given by $x^d = 100 - p + \frac{dp}{dt}$ and $x^s = -50 + 2p + 10\frac{dp}{dt}$. Find the time path of p for dynamic equilibrium if initial price is given to be ` 20. What will be the price at time t = 10?

Solution : From the demand-supply equality, we get, $-50 + 2p + 10\frac{dp}{dt} = 100 - p + \frac{dp}{dt}$

or, $9\frac{dp}{dt} + 3p = 150$ or, $\frac{dp}{dt} + \frac{1}{3}p = \frac{50}{3}$.

This is a first order differential equation of the standard from : $\frac{dy}{dt} + ay = b$.

The solution of this equation is :

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$$

In our context, $a = \frac{1}{3}$, $b = \frac{50}{3}$.

$\therefore$ p(t) =

$\therefore$ $p(t) = [20 - 50]e^{-t/3} + 50$ as $p(0) = 20$

$\therefore$ $p(t) = 50 - 30e^{-t/3}$ is our time path of price.

As $t \to \propto$, $e^{-t/3} \to 0$. So, $p(t) \to 50$

This is the long run equilibrium price.

Now, if t = 10, $p(10) = 50 - 30\, e^{-10/3}$ **(Ans.)**

6.7 Some Problems on Dynamic Analysis with Solutions

Example 6.14 : Examine whether the market is stable if $D_t = 30 - 5p_t$ and $S_t = 20 - p_{t-1}$.

Solution : Equating $D_t = S_t$, we get, $30 - 5p_t = 20 - p_{t-1}$

or, $5p_t = p_{t-1} + 10 \quad \therefore p_t = \frac{1}{5}.p_{t-1} + 2.$

This is a first order non-homogeneous difference equation.

Its solution is : $p_t = (p_0 - \bar{p})\left(\frac{1}{5}\right)^t + \bar{p}$ where $\bar{p}$ is the equilibrium price.

Setting $p_t = p_{t-1} = \bar{p}$, we get, $5\bar{p} = \bar{p} + 10$ or, $4\bar{p} = 10 \quad \therefore \bar{p} = 2.5$

So, the solution is : $p_t = (p_0 - 2.5)\left(\frac{1}{5}\right)^t + 2.5$

As $t \to \infty$, $\left(\frac{1}{5}\right)^t \to 0$. So, $p_t \to 2.5$

Thus the equilibrium is stable. This can be shown mathematically.

Putting $D_t = S_t$, $p_t = \frac{1}{5}p_{t-1} + 2$

This is a first order non-homogeneous difference equation. To get its solution, we first consider the homogeneous part : $p_t = \frac{1}{5}p_{t-1}$.

Let $p_t = x^t$ be a solution. Then $x^t = \frac{1}{5}.x^{t-1}$

$\therefore x = \frac{1}{5}$ (assuming $x^{t-1} \neq 0$)

$\therefore p_t = \left(\frac{1}{5}\right)^t$ is the solution of the homogeneous part.

Then $p_t = H\left(\frac{1}{5}\right)^t$ is also a solution where the value of H is to be determined from the initial condition (putting t = 0).

For getting particular solution, we put, $p_t = p_{t-1} = \bar{p}$.

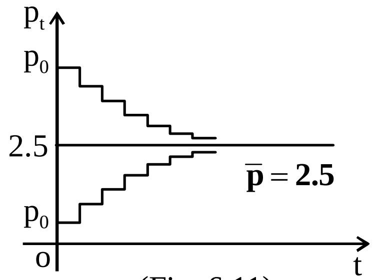
Then we have, $\bar{p} = \frac{1}{5}\bar{p} + 2 \;\; \therefore \;\; \frac{4}{5}\bar{p} = 2 \quad \therefore \;\; \bar{p} = \frac{2 \times 5}{4} = \frac{5}{2} = 2.5$.

So, we have $p_t = H\left(\frac{1}{5}\right)^t + \frac{5}{2}$.

Putting t = 0, we get, $p_0 = H + \frac{5}{2}$ $\therefore H = p_0 - 2.5$

So, time path of p is : $p_t = (p_0 - 2.5)\left(\frac{1}{5}\right)^t + 2.5$

As $t \to \infty$, $p_t \to 2.5$. If initial price p_0 is less than 2.5, it will gradually rise to 2.5. If initial price p_0 is greater than 2.5, it will gradually come down to 2.5.



(Fig. 6.11)

This is shown in our figure 6.11.

Example 6.15 : Consider the stability of equilibrium in the following dynamic model :

(i) $Q_d = 100 - 10p$

(ii) $Q_s = 25 + 15p$

(iii) $\frac{dp}{dt} = 0.10\ (Q_d - Q_s)$

Solution : Substituting (i) and (ii) into equation (iii),

we get, $\frac{dp}{dt} = 0.10(100 - 10p - 25 - 15p) = 0.10(75 - 25p)$

$\therefore \frac{dp}{dt} + 2.5p = 7.5.$

This is a first order non-homogeneous differential equation of the standard form : $\frac{dy}{dt} + ay = b$. The solution of this equation is : $y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$.

In our context, a = 2.5, b = 7.5. So, $\frac{b}{a} = \frac{7.5}{2.5} = 3$

Hence the solution of the given system is : $p(t) = [p(0) - \bar{p}]e^{-2.5t} + \bar{p}$

Here $\bar{p} = \frac{b}{a} = 3$.

Alternatively, $\bar{p}$ is the equilibrium price obtained by setting $Q_d = Q_s$, i.e., $100 - 10\bar{p} = 25 + 15\bar{p}$.

or, $25\bar{p} = 100 - 25 = 75 \therefore \bar{p} = 3$.

Or, setting $\frac{dp}{dt} = 0$, we get equilibrium p.

i.e., $2.5p = 7.5 \therefore$ Equilibrium $p = \bar{p} = \frac{7.5}{2.5} = 3$

Thus, the time path of price is :

$p(t) = [p(0) - 3]\, e^{-2.5t} + 3$

As $t \to \propto$, $e^{-2.5t} \to 0 \therefore p(t) = 3$

Our equilibrium is stable. Figure 6.10 represents similar idea.

Example 6.16 : Given $Y(t) = bK(t)$, $S(t) = s.Y(t)$.

Find the equilibrium time path of Y.

Solution : $Y(t) = bK(t) \therefore K(t) = \frac{1}{b} Y(t)$

So, $I(t) = \frac{dK(t)}{dt} = \frac{1}{b} \cdot \frac{dY(t)}{dt}$

Again, in equilibrium, $S(t) = I(t) \therefore sY(t) = \frac{1}{b} \cdot \frac{dY(t)}{dt}$

or, $\frac{\frac{dY(t)}{dt}}{Y(t)} = sb$ or, $\frac{dY(t)}{Y(t)} = sbdt$

Integrating we get, $\log Y(t) = sbt + \log c$ where $\log c$ = constant.

or, $\log\left(\frac{Y(t)}{c}\right) = \log e^{sbt} \therefore \frac{Y(t)}{c} = e^{sbt} \quad \therefore Y(t) = c.e^{sbt}$

When t = 0, Y(0) = c

So, $Y(t) = Y(0)e^{sbt}$

This is the time path of Y.

Example 6.17 : Given that $Q_t^d = 120 - 0.5\, p_t$, $Q_t^s = -30 + 0.3\, p_t$ and

$p_{t+1} = p_t - 0.2\left(Q_t^s - Q_t^d\right)$ and $p_0 = 200$. Find the time path of price (p).

Ans. $p_t = 12.5(0.84)^t + 187.5$

Example 6.18 : Examine the dynamic stability of the equation :

$Y_{t+2} - 11Y_{t+1} + 10Y_t + 27 = 0$.

Solution : We have, $Y_{t+2} - 11Y_{t+1} + 10Y_t + 27 = 0$. This is a second-order non-homogeneous difference equation. We first consider the particular solution. Here $Y_{t+2} = Y_{t+1} = Y_t = \bar{Y}$ will not yield any particular solution. So we try $Y_t = \bar{Y}.t$ as a particular solution. Hence, $\bar{Y}(t + 2) - 11\bar{Y}(t + 1) + 10\bar{Y}t + 27 = 0$

or, $\bar{Y}t + 2\bar{Y} - 11\bar{Y}t - 11\bar{Y} + 10\bar{Y}t + 27 = 0$

or, $-9\bar{Y} + 27 = 0$, or, $9\bar{Y} = 27$ or, $\bar{Y} = \frac{27}{9} = 3$ is the particular solution.

We now consider the solution of the homogeneous part. Let $Y_t = Hb^t$ be the trial solution.

So, $Hb^{t+2} - 11Hb^{t+1} + 10Hb^t = 0$

$\therefore b^2 - 11b + 10 = 0$, (assuming $H \neq 0$ and $b^t \neq 0$)

or, $b^2 - 10b - b + 10 = 0$

or, $b(b - 10) - (b - 10) = 0$

or, $(b - 10)(b - 1) = 0$

$\therefore b = 1, 10$ i.e., $(b_1, b_2) = 1, 10$

The solution of the homogeneous part is : $Y_t = \beta_1 b_1^t + \beta_2 b_2^t$ where b_1 and b_2 are the two roots of the quadratic $b^2 - 11b + 10 = 0$ and the values of β_1 and β_2 are determined from the initial condition. So, the time path of Y is : $Y_t = \beta_1(1)^t + \beta_2(10)^t + \bar{Y}$

or, $Y_t = \beta_1 + \beta_2(10)^t + 3$.

As $t \rightarrow \propto$, $Y_t \rightarrow \propto$. So the time path is unstable.

Example 6.19 : Find the solution of the equation $Y_t = 10Y_{t-1} - 16Y_{t-2} + 14$, given

$Y_0 = 10$ and $Y_1 = 36$.

Solution : The given equation $Y_t = 10Y_{t-1} - 16Y_{t-2} + 14$ is a second order linear non-homogeneous difference equation. Let the particular solution be :

$Y_t = Y_{t-1} = Y_{t-2} = \bar{Y}$ $\therefore$ $\bar{Y} = 10\bar{Y} - 16\bar{Y} + 14$ or, $17\bar{Y} - 10\bar{Y} = 14$

or, $7\bar{Y} = 14$ $\therefore$ $\bar{Y} = \frac{14}{7} = 2$.

Now we consider the solution of the homogeneous part : $Y_t = 10Y_{t-1} - 16Y_{t-2}$.

Let $Y_t = Hb^t$ be the trial solution.

$\therefore Hb^t - 10Hb^{t-1} + 16Hb^{t-2} = 0$

or, $b^2 - 10b + 16 = 0$ (assuming $H \neq 0$ and $b^{t-2} \neq 0$)

or, $b^2 - 2b - 8b + 16 = 0$

or, $b(b - 2) - 8(b - 2) = 0$, or, $(b - 2)(b - 8) = 0$

$\therefore b = 2, 8$, i.e., $(b_1, b_2) = (2, 8)$.

So, the complete solution is :

$Y_t = \beta_1(2)^t + \beta_2(8)^t + 2$

The values of β_1 and β_2 will be known from the initial conditions. Putting t = 0, we have, $\beta_1 + \beta_2 + 2 = Y_0 = 10$ given

$\therefore\ \beta_1 + \beta_2 = 8.$

Again, if t = 1, $2\beta_1 + 8\beta_2 + 2 = Y_1 = 36$ or, $2\beta_1 + 8\beta_2 = 34$

$\therefore\ \beta_1 + 4\beta_2 = 17$

Solving these two equations we get, $\beta_1 = 5$ and $\beta_2 = 3$.

So the time path of Y is : $Y_t = \beta_1(2)^t + \beta_2(8)^t + 2$

or, $Y_t = 5(2)^t + 3(8)^t + 2$

As $t \to \propto$, $Y_t \to \propto$. So, the time path is unstable.

Example 6.20 : $D_t = 19 - 6p_t$ and $S_t = -5 + 6p_{t-1}$. Find the equilibrium price and the time path of price. Is the equilibrium stable 2?

Solution : Putting $D_t = S_t$, we have, $19 - 6p_t = -5 + 6p_{t-1}$. or $6p_t + 6p_{t-1} = 24$, or, $p_t + p_{t-1} = 4$. This is a first order linear non-homogeneous difference equation. Solving it, we shall get the time path of p.

Putting $p_t = p_{t-1} = \bar{p}$, we get the equilibrium price or the particular solution. Thus $\bar{p} + \bar{p} = 4.\ \therefore\ \bar{p} = 2$

We now consider the solution of the homogeneous part : $p_t + p_{t-1} = 0$, or, $p_t = -p_{t-1}$.

Let $p_t = Hb^t$ be the trial solution. Then, $Hb^t = -Hb^{t-1}$

$\therefore\ b = -1$(assuming $H \neq 0$ and $b^{t-1} \neq 0$).

Hence, complete solution is : $p_t = H(-1)^t + 2$. Putting t = 0, we get the value of H. Thus, $p_0 = H + 2\ \therefore\ H = p_0 - 2$. Thus, total solution is : $p_t = (p_0 - 2)(-1)^t + 2$.

If t is odd, $(-1)^t < 0$. If t is even, $(-1)^t > 0$. So the time path will have oscillations. Further, $|-1| = 1$.

So, there will be constant oscillations. (Readers may refer to the figure 6.7 in which we have shown constant oscillations of price).

6.8 A Note on Dynamic Optimisation

Optimum means the best situation or state of affairs. To achieve an optimum is to optimise and a situation which is an optimum, is said to be optimal. So, optimisation means the process or technique of achieving an optimal situation. When we optimise something, we want to maximise or minimise something. For example, a consumer wants to maximise utility; a firm wants to maximise profit or to minimise cost, etc. In unit 3, we have considered the problem of maximisation or minimisation of a variable without constraint. Next we have analysed the problem of optimisation (i.e., maximisation or minimisation) with constraint. For example, we have deduced the conditions of utility maximisation subject to budget constraint, or cost minimisation subject to an output constraint, etc.

But those treatments were static optimisation. There we tried to find out a single value for each choice variable, such that a stated objective function was maximised or minimised. Such a process has no time dimension. In contrast, we introduce time explicitly in a dynamic optimisation problem. In such a problem, we have planning period from an initial time t = 0 to a terminal time t = T. Here we try to find the best course of action during that entire period. Thus, the solution for any variable takes the form of not a single value, but a complete time path.

The classical approach to dynamic optimisation is called the calculus of variation. Later, a more powerful approach gradually developed. It is now known as optimal control theory which replaced the calculus of variation. It uses the maximum principle to achieve dynamic optimisation.

We may present a standard form of optimum control theory of dynamic optimisation. Suppose we want to maximise profit over a time period. At any point of time t, we have to choose the value of some control variable, u(t). It will then affect the value of some state variable, y(t), via a so-called equation of motion.

In turn, y(t) will determine the profit $\pi(t)$. Our objective is to maximise the profit over the entire period (0 – T). Hence the objective function should take the form of a definite integral of π from t = 0 to t = T. The problem also specifies the initial value of the state variable y, say, y(0) and the terminal value of y, say, y(T). In other words, the model specifies the range of values which y(T) is allowed to take.

Now we may state the simplest problem of optimal control as follows :

Maximise $\int_0^T F(t, y, u)dt$ ...(1)

subject to $\frac{dy}{dt} (\equiv y') = f(t, y, u)$ ...(2)

$y(0) = A$ $\quad$ $y(T)$ free ...(3)

and $u(t) \in U$ for all $t \in (0, T)$ [$\in$ implies belongs to] ...(4)

Equation (1) is our objective function. It shows how the choice of control variable u at time t, along with the resulting y at time t, determines our object of maximisation at t. Equation(2) is the equation of motion for the state variable y. It provides the mechanism by which our choice of control variable u can be translated into a specific pattern of movement of the state variable y. Equation(3) states that in the initial state, the value of y at t = 0, is a constant A, but the terminal state Y(T) is left unrestricted. Finally, our equation (4) states that the permissible choices of u are limited to a control region U. However, it may also happen that u(t) is not restricted.

6.9 Summary

1. Use of Difference Equation : Difference equation is used when time is taken as discrete variable. This equation gives us the equilibrium value of a variable and also the rate of change of the variable over time.

2. Solution of a Difference Equation : Solution of a difference equation has two components : complementary solution and particualr solution. The complementary solution gives the nature of time path of a variable while the particular solution gives the equilibrium value of variable.

3. Differential Equation : Differential equation is used when time is treated as a continuous variable. The solution of a differential equation has also two parts : complementary solution and particular solution. White the particular solution gives the long run equilibrium value of a variable, the complementary solution informs us ablut the nature of time path of the variable.

4. Application Difference Equation in Economics : Difference Equation has many uses in Economics. In particular, with the help of difference equation, we may dicuss Keynesion dynamic multiplier, cobweb model of price variations and multiplier accelerator model of trade cycle.

5. Application of Differential Equation in Economics : Differential equation has also many applications. In particular, with the help of differential equation, we may discuss Domar model of economic growth and price dynamics in a competitive model.

6.1 Excercises

Short Answer Type Questions

1. What is static analysis?
2. What do you mean by comparative static analysis?
3. What is dynamic analysis?
4. Give the general form of a linear non-homogeneous difference equation.
5. What is linear homogeneous difference equation?

6. Give the general form of a first order linear non-homogeneous difference equation.
7. Write down a first order linear homogeneous difference equation.
8. What are the components of total solution of a difference equation?
9. What is the nature of time path of Y if $Y_t = 4(2.5)^t + 10$?
10. What kind of time path is represented by $Y_t = 4(-0.5)^t + 20$?
11. State the nature of time path of Y if $Y_t = 5(-1)^t + 30$.
12. What is differential equation?
13. When is a differential equation used in a dynamic analysis?
14. When is a difference equation used in a dynamic analysis?
15. What is a cobweb model?

Medium Answer Type Questions

1. Distinguish among static analysis, comparative analysis and dynamic analysis.
2. Why is dynamic analysis necessary?
3. What is iterative method of solving a difference equation?
4. How can dynamic analysis deal with the Keynesian static multiplier when b = 1?
5. How will you treat the Keynesian static multiplier $\left(=\frac{1}{1-b}\right)$ when b > 1?
6. Mention some limitations of multiplier-accelerator model of trade cycle.
7. Solve the difference equation, $\Delta Y_t = 0.5Y_t$, given $Y = Y_0$ when t = 0.
8. Solve the difference equation $Y_t - Y_{t-1} = 3Y_{t-1}$, given $Y_t = Y_0$ when t = 0
9. $Y_t = Y_{t-1} + 6$, given initial income = Y_0. Solve the equation.
10. At t = 0, $Y_t = Y_0$. Now solve the difference equation $Y_t - Y_{t-1} = 0$
11. If $Y = Y_0$ at t = 0, the deduce the time path of Y of the equation $Y_t - 2Y_{t-1} = 0$
12. Given $C_t = 200 + 0.75\ Y_{t-1}$, $I_t = 50 + 0.15Y_{t-1}$ and $Y_0 = 3000$, find time path of Y. Is the equilibrium stable?
13. Write a short note on dynamic optimisation.

Long Answer Type Questions

1. Describe the process of solution of a first order linear difference equation.
2. Describible with suitable examples the process of solution of difference equations by

iterative method.

3. Consider the problem of dynamic stability of equilibrium of the time path $Y_t = Ab^t + Y_0$ taking different values of b. (A = constant).
4. Describe the process of solution of a second order non-homogeneous difference equation.
5. How will you solve a first order linear differential equation?
6. Explain how Keynesian multiplier theory can be dynamised and its inconsistencies can be tackled when $b \geq 1$.
7. Describe the cobweb model of price fluctuation taking first order difference equations of demand and supply functions.
8. Explain the multiplier accelerator model of trade cycle as formulated by Samuelson.
9. Analyse the Domar Model of economic growth in order to explain the concept of knife edge instability.
10. Using differential equations, describe the process of price dynamics in a competitive model.
11. $D_t = 18 - 3p_t$, $S_t = -3 + 4p_{t-1}$. Is the equilibrium stable?
12. Consider the following multiplier-accelerator model : $C_t = \alpha Y_{t-1}$, $I_t = \beta(C_t - C_{t-1})$ and $Y_t = C_t + I_t$. Here, $\alpha = 0.9$ and $\beta = 0.5$. Find the time path of income (Y) and examine the nature of the time path.

6.11 References

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